Notes on Perspective Viewing Transformations

Michael J. Scott

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We consider how to view an object, which is described by a vector list, from an arbitrary point in space (and looking at the origin of the world coordinate frame). We shall use the world coordinate frame as the object viewed; thus the simple vector list looks something like this:

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0 0 0
0 0 1
0 0 0
0 1 0
0 0 0
1 0 0
```

Our world coordinate frame is a little fancier, having simple labels, a square 0.5 units on each coordinate plane, and a tick mark at $(0, 0.5, 0.25)$. We will follow this point in its homogeneous coordinate row vector representation $p = (0, 0.5, 0.25, 1)$ through the transformations. See Figure 1. The labels -1 to 1 on the outside frame indicate the screen coordinates of the display and have nothing to do with the viewing transformations discussed here. The letters X, Y, and Z at 0.5 on each axis belong to the eye coordinate axis which begins coincident with the world coordinate axis.

Say we want to view the object (in this case, the triad of world coordinate axes) from the location (a, b, c). In order to put the origin of the eye coordinates at (a, b, c), we translate the object by $(-a, -b, -c)$ using the translation matrix $T_1 = T_{(-a,-b,-c)}$:

$$T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -a & -b & -c & 1 \end{pmatrix}$$

Figure 2 shows the result of this transformation. The axes representing the eye have moved to $(3, 5, 2)$. Note that $pT_1 = (-3, -4.5, -1.75, 1)$ is the location of our original point p with respect to the eye frame. As a second check, the world origin resides at (-3,-5,-2) in the eye frame.

The next step is to rotate the eye frame by $+90°$ about its own x-axis so that the z_{eye}-axis is parallel to the $x_{\text{world}} - y_{\text{world}}$ plane. This is accomplished with a $-90°$ rotation $R_1 = R_{x,-90}$:

$$R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

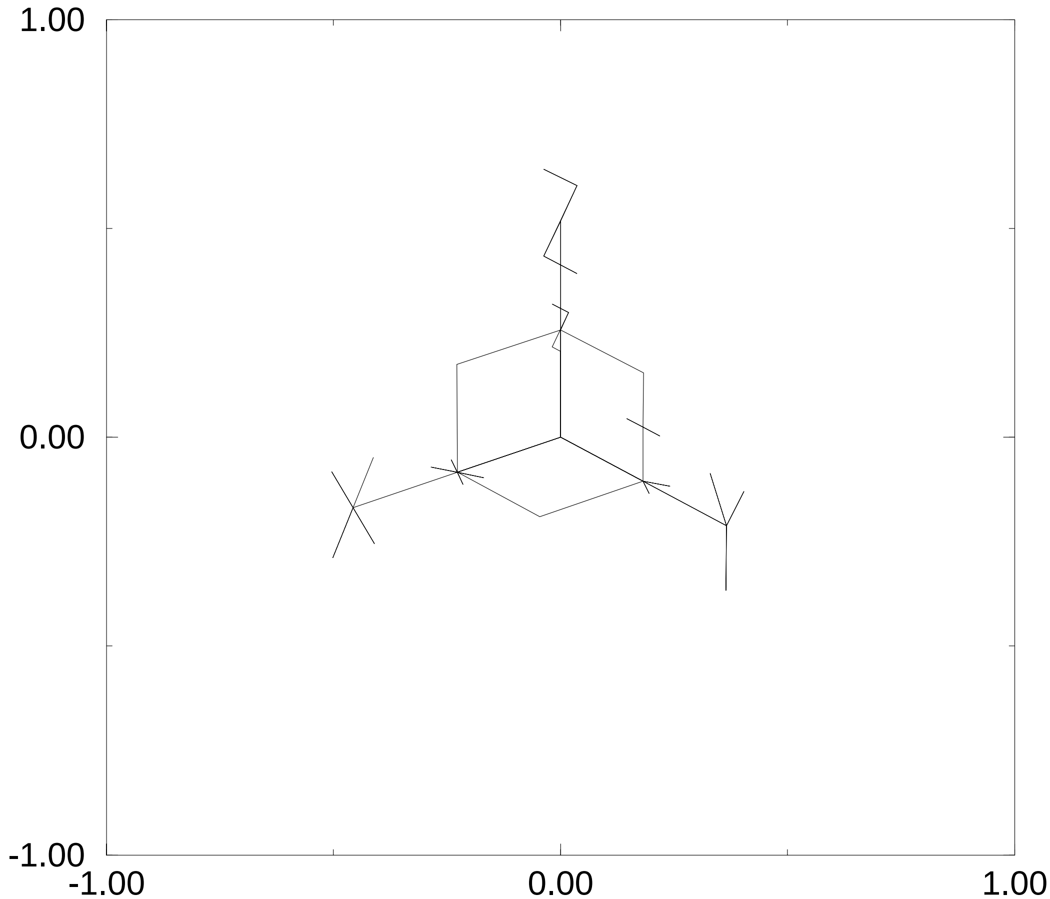

Figure 1: World Coordinate Axes.

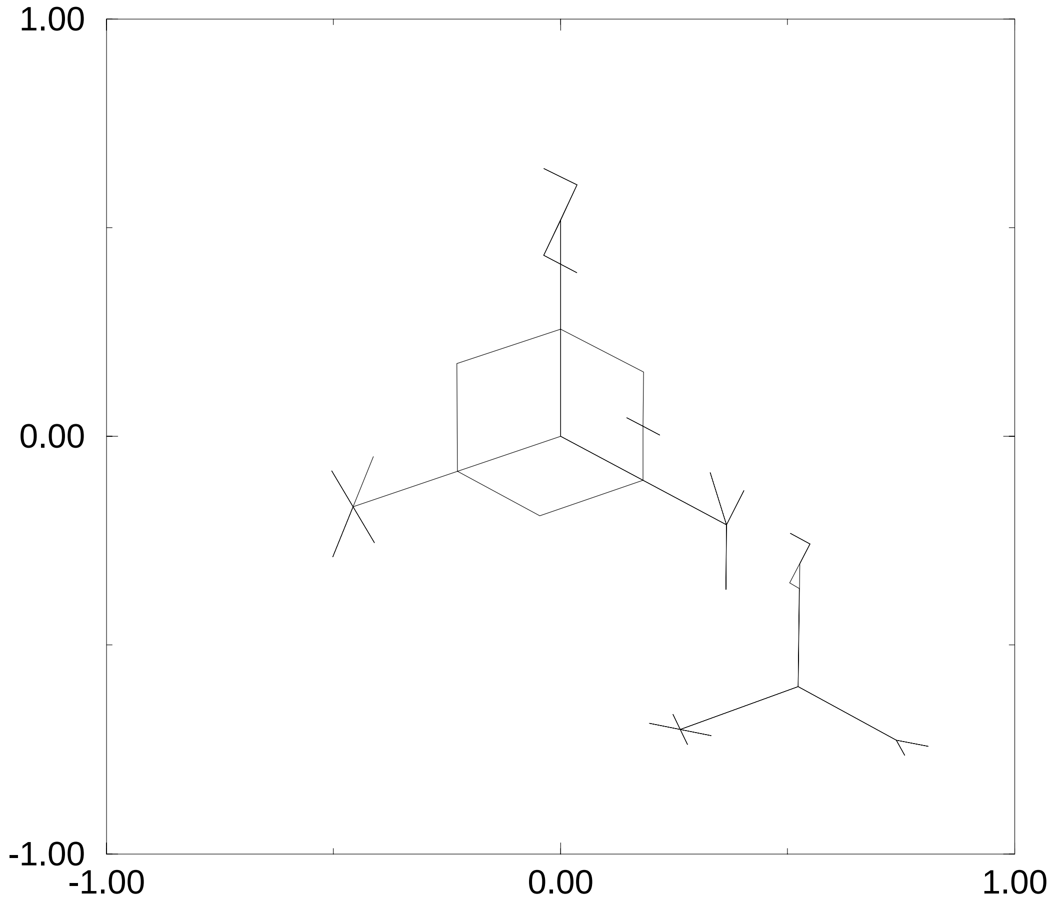

Figure 2: Step 1: Translate $(-a, -b, -c)$.

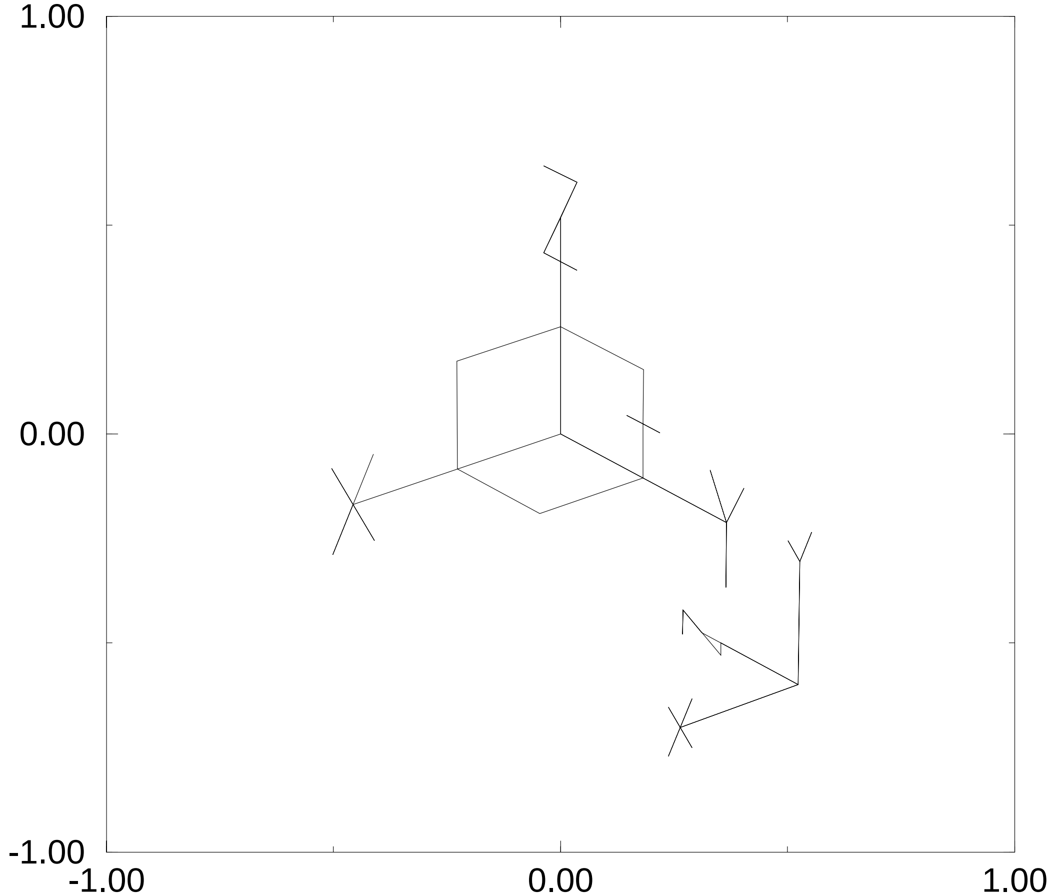

Figure 3: Step 2: Rotate $-90°$ about x.

(It's easy to tell that we have the correct negative sign; a rotation of $-90°$ about x should move z-axis (0,0,1,1) to point in the y direction (0,1,0,1), and it does.) We see now that

$$pT_1R_1 = (-3, -1.75, 4.51)$$

when seen in the eye frame. See Figure 3. The world origin has moved to (-3,-2,5) in the eye frame.

This rotation leaves the y_{eye}-axis pointing straight up, which is where we want it when we need it to define the viewing screen.

Now we rotate about the y_{eye}-axis until the z_{eye}-axis points straight at the z_{world}-axis. If $\theta = \tan^{-1}(a/b)$, then the eye frame must rotate through $-\theta$ about the

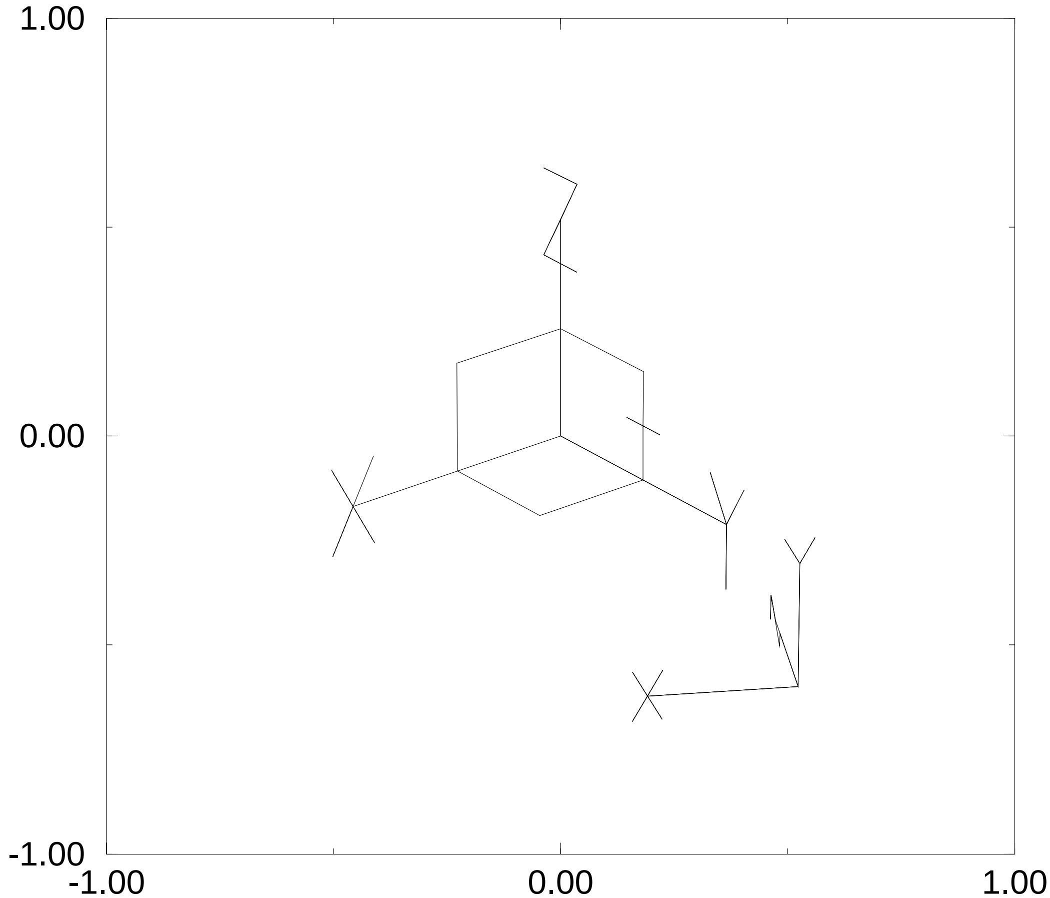

Figure 4: Step 3: Rotate by θ about y.

y-axis. Of course this is accomplished with the inverse rotation $R_2 = R_{y,\theta}$:

$$R_2 = \begin{pmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Figure 4 shows this development. You can check that the locations of p and the world origin.

The next rotation should take the z_{eye}-axis to point directly at the origin of the world frame. This is a positive rotation about the x_{eye}-axis through an angle ψ equal to:

$$\psi = \tan^{-1}\left(\frac{c}{\sqrt{a^2+b^2}}\right)$$

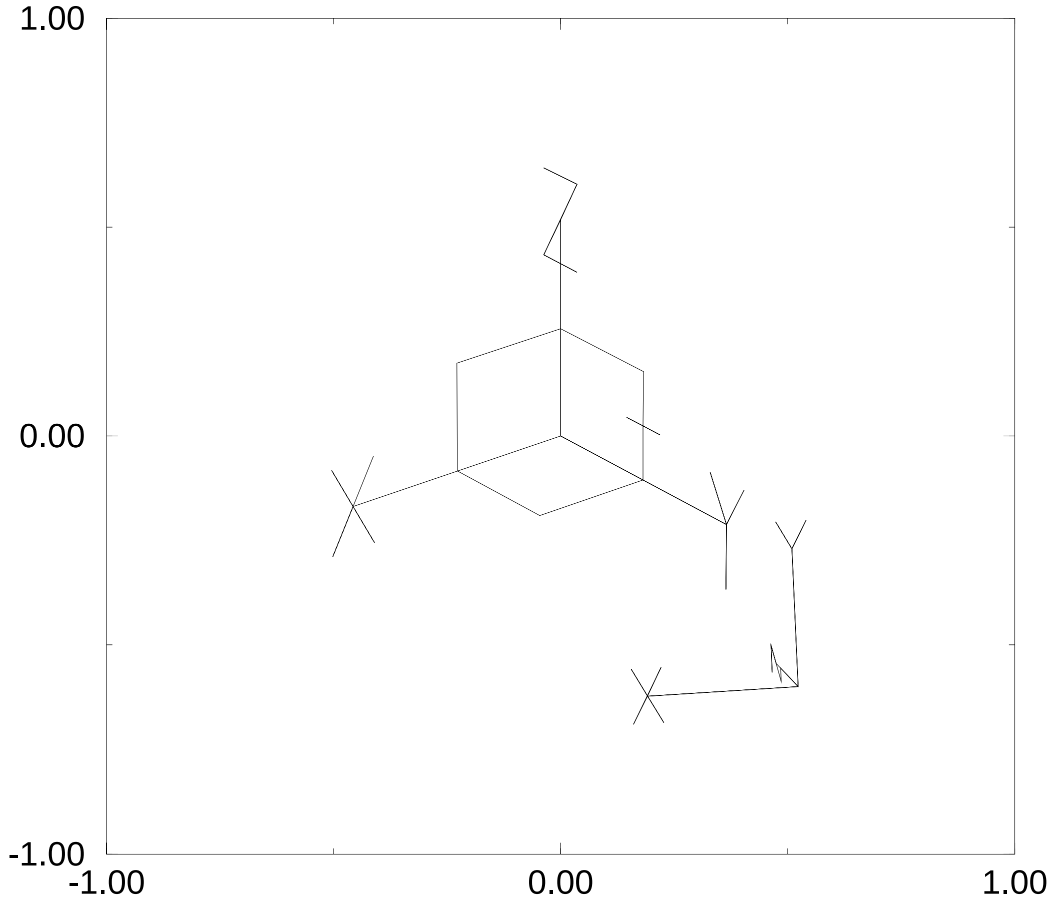

Figure 5: Step 4: Rotate by $-\psi$ about x. The z_{eye}-axis now points to the world origin.

The necessary rotation matrix, R_3, is naturally the negative rotation about x:

$$R_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi & 0 \\ 0 & \sin\psi & \cos\psi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The situation after all of these moves is shown in Figure 5. Check that the world origin ends up in the right place with respect to the eye frame:

$$(0, 0, 0, 1)T_1R_1R_2R_3 = (0, 0, \sqrt{a^2 + b^2 + c^2}, 1)$$

The only remaining transformation is to change the eye frame to a left-handed coordinate system, so that the observer sitting on the eye frame and looking at the world origin sees an x-axis pointing right, a y-axis pointing up, and a z-axis pointing forward. We accomplish this with an unorthodox but intuitively appealling transformation to flip the x_{eye}-axis to the other side:

$$V = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The finally transformed axes are shown in Figure 6. By multiplying the entire vector list by $T_1 R_1 R_2 R_3 V$ we have prepared it for projection onto a screen for a perspective view. The details of that transformation are not covered here.

If you want to be more orthodox and flip the z-axis rather than the x-axis, you can accomplish that by replacing $R_2 = R_{y,\theta}$ with $R_2' = R_{y,\theta+180^\circ}$. The z_{eye}-axis will then point away from the world origin, the x_{eye}-axis will point to the right, and V' which flips the z-axis will put the frames in the same ending position that we have reached.

While the details of the projection onto a screen are not shown here, Figure 7 shows the same world and eye frames as were seen in Figure 6, but this time looking straight down the z_{eye}-axis, from far enough away that both frames are visible.

Food for thought (extra credit): What viewing transformations were applied to this whole “universe” (both eye and world frames) in order to show the steps of moving the eye frame to its final vantage point?

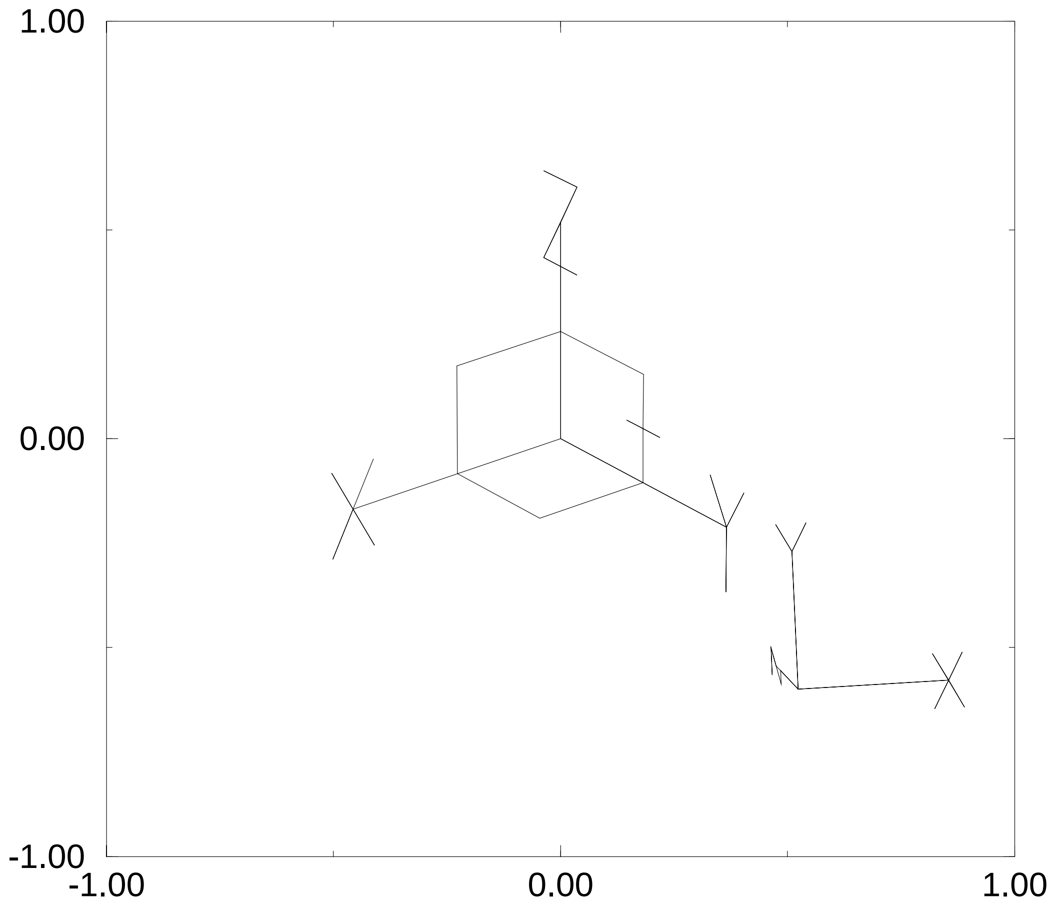

Figure 6: Step 5: Change to left-handed coordinates.

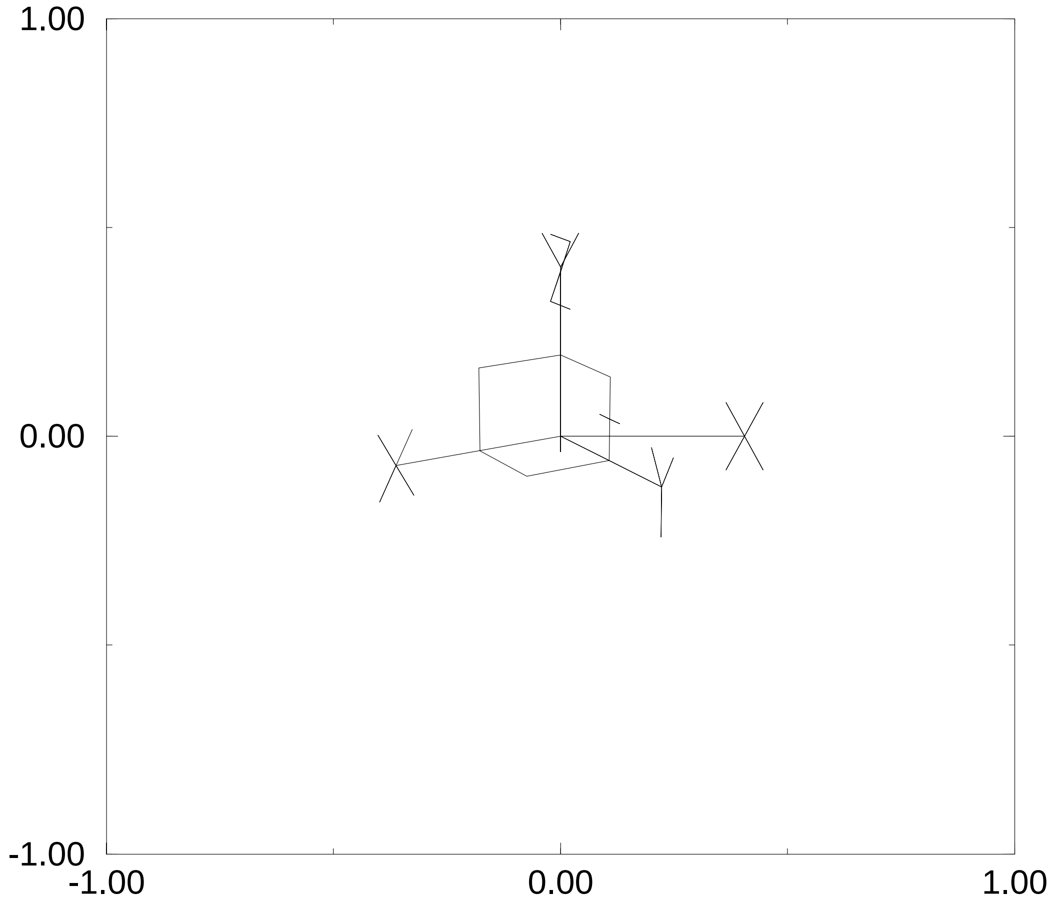

Figure 7: Final position, shown looking down z_{eye}-axis.