Chapter 3

DESIGN PARAMETER SELECTION IN THE PRESENCE OF NOISE

Kevin N. Otto and Erik K. Antonsson

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Abstract

The *method of imprecision* is a design method whereby a multi-objective design problem is resolved by maximizing the overall degree of *designer preference*: values are iteratively selected based on combining the degree of preference placed on them. Consider, however, design problems that exhibit multiple uncertainty forms (noise). In addition to degrees of preference (*imprecision*) there are *probabilistic* uncertainties caused by, for example, measuring and fabrication limitations. There are also parameters that can take on any value *possible* within a specified range, such as a manufacturing or tuning adjustment. Finally, there may be parameters which must *necessarily* satisfy all values within the range over which they vary, such as a horsepower requirement over a motor's different speeds. This paper defines a "best" set of design parameters for design problems with such multiple uncertainty forms and requirements.

1. Introduction

Much of the process of formalizing design decision making has yet to be understood. Rigorous methods for representing and manipulating the concerns and unknowns of a design need to be advanced for a robust automation. The ability to make such decisions, however, is hampered by uncertainties (noise) in the variables used, both in their values and in the manner in which they should be manipulated. This paper will introduce methods that appropriately model the various forms of uncertainty to define an overall "best" set of parameters for use.

Parameter types are broken down by characterizing some model variables as *design parameters*, or those parameters in the engineering model for which the designer must select values, such as geometric sizes. *Performance parameters* are those the designer uses to indicate a design's ability to satisfy *functional requirements*, such as stress or deflection. Finally, there are usually *noise parameters* that introduce uncertainty in the designer's ability to measure, such as manufacturing errors. Note this use of the term *parameter* is therefore different from its usage in statistics, where it is used to describe a entire set of noise values in some respect, usually with a single number (*e.g.*, an expected value).

Initially, design parameter values are uncertain: the designer does not know what values to use. Consequently, the performance parameter values are also uncertain. As a design process proceeds, values are determined more and more precisely in an iterative test and refine fashion. Noise parameter uncertainty, however, always exists, and requires changes in measuring and manufacturing processes to improve.

In addition to these basic uncertainties, there are several different types of design parameters [21]. Some parameters may have absolute, rigid functional requirements. Others may be flexible; targets only express what is desired, not required. Similarly, tolerances may have strict limits placed on them; failure to be within the tolerances is a failed design. Other tolerances may be flexible. This paper introduces a modeling scheme for both parameter types, and defines methods for selecting an overall best design parameter set with such influences. Those parameters that have strict requirements are termed *necessary parameters*, and may be of the design, performance, or noise variety. The mathematics of necessity was introduced to engineering design by Ward and Seering [20] for interval mathematics. The concept is extended here to different uncertainty forms.

Modeling of Uncertainty

Every uncertainty form discussed above shall be directly modeled. That is, the initial design parameter uncertainty shall be modeled using *the method of imprecision* [8, 22] where each design parameter value is given a rank from zero to one to indicate degree of preference. This forms a preference function μ over each design parameter and performance parameter indicating degree of preference for values. Co-dependencies are possible, see [8].

Similarly, probabilistic noise parameters [10] shall have their values ranked with degrees of probability. Finally, possibilistic noise parameters [10] shall have their values ranked with degrees of possibility. All these uncertainties reflect different phenomena, and consequently will have different derived mathematics. Of course, these uncertainties can interact. A design parameter may have both a preference function and a probability density function.

This paper focuses on making the determination of the "best" overall design parameter set given such influences, and presents a method which can be used to solve for the maximum overall preference for a design parameter set, even with confounding probabilistic and possibilistic uncertainties, as well as necessary requirements.

Section 2.1 briefly reviews a metric to define the highest preference, see [8] for a complete discussion of its iterative specification. The effects of uncertainty on the highest overall preference concept are discussed in Section 2.2 for individual forms of confounding influences. Section 3. discusses incorporation of necessary parameters. Finally, Section 4. discusses design problems with combinations of these effects.

2. Parameter Selection in the Design Process

2.1 Global Preference Functions and Design Strategies

This work addresses the stage of the engineering design process where the designer is selecting a configuration. Therefore, the designer has developed a formal design parameter space (DPS) consisting of alternative configurations to choose among. The DPS will be characterized by *design parameters*, $d_1, \ldots, d_n$. For a design process of selection among alternatives, each d_i represents an alternative, so the DPS $\simeq \mathbb{Z}_n$ (the finite set of integers up to n). For determination of values in a design model, there are usually multiple parameters each of which could be thought of as a continuum, and so each d_i might thought of as a value of a vector within a DPS $\simeq \mathbb{R}^n$, where the DPS is represented in some basis with coordinates d^{i}, for example.

Given that an overall best design parameter set is to be found, the "best" concept must be defined. Unfortunately, the various performance parameters usually involve incommensurate concepts. A traditional approach to combining incommensurate parameters is to use a normalization and a weighting. Instead, incommensurate parameters can be combined more usefully using a common trait: designer preference. This means that preference information (μ) on the design parameters (d^i) and requirement preferences on the the performance parameters ($f_j(\vec{d})$) must be combined into an overall preference rating for that design parameter set ($\vec{d}$). The vector notation is meant to suggest the typical design scenario of a multi-component design, but this development is applicable to singleton designs.

To reflect the designer's overall preference, a global *design metric* (which ranks each combination of possible design parameter arrangements) must exist across the design parameter space, expressed as a function of the known preferences of the design:

$$\mu(\vec{d}) = \mathcal{P}\left(\mu(d^1), \ldots, \mu(d^n), \mu(f_1(\vec{d})), \ldots, \mu(f_q(\vec{d}))\right) \tag{3.1}$$

where $\vec{d}$ is a design parameter arrangement, and f_1 through f_q are the performance parameters. This definition is the least possibly restrictive method of combining incommensurate concepts. It is a formalization of the notion of combining incommensurate parameters based on the degree that each parameter satisfies the designer. Hence the best design parameter set to use is defined by the maximum of this preference function:

$$\mu(\vec{d^*}) = \sup\left\{\mathcal{P}(\mu(d^1), \ldots, \mu(d^n), \mu(f_1(\vec{d})), \ldots, \mu(f_q(\vec{d}))) \;\middle|\; \vec{d} \in \text{DPS}\right\} \tag{3.2}$$

The multi-objective design problem with multiple constraints then becomes to find $\vec{d^*}$ as reflected in Equation 3.2, by maximizing the design metric $\mathcal{P}$ over the design space (DPS).

This problem statement, however, is incomplete: $\mathcal{P}$ is unspecified. The choice of a *trade-off strategy* desired to be used by the designer will specify $\mathcal{P}$. Formal trade-off strategies for engineering design are introduced in [8]. For example, use of the minimum function ($\mathcal{P} \equiv$ *min*) reflects a *conservative* or *non-compensating* strategy of the designer to improve the design by always improving the weakest design aspect. Use of a normalized multiplication reflects an *aggressive* or *compensating* strategy of the designer to develop a maximally performing design. Of course, non-compensating and compensating strategy hybridization for different aspects of the design are possible [8].

2.2 Uncertainty Effects

Having formulated the overall preference function, there may still be uncertainties to confound the search for the design parameter set which provide the highest global preference. This will be resolved, however, by assuming that the designer wants the design parameter set that provides the best "quality" in light of the possible variations, using the following interpretation of quality: overall preference despite variations. This is similar to the view of quality that Taguchi uses [17], where he instead applies the view on a single performance parameter, rather than on preference over many parameters [9].

Confounding Influences

In an engineering design problem, noise is typically characterized by *noise parameters*, $n_1, \ldots, n_k, \ldots, n_q$. A noise parameter n_k might be the possible positioning of an operator switch, and so the alternatives may be finite. Alternatively, n_k might be a value of a manufacturing error on a design parameter, and so the NPS may have a continuum of possibilities. In this case, n_k might be thought of as a value of a vector within an NPS $\simeq \mathbb{R}^q$ represented with coordinates n^κ in some basis, for example.

Though these specific, simple examples illustrate the concept of noise, the structure needed to model noise can be developed more generally. Thus, the NPS is defined to be an *uncertainty measure space (NPS, $\mathcal{B}$, g)*, which is a set NPS of elements n, a σ-algebra $\mathcal{B}$ of sets over the NPS (also called a *Borel field*), and an uncertainty measure $g : \mathcal{B} \rightarrow [0, 1]$. Notice that $\mathcal{B}$ is a set of sets. $N \in B$ is an event: a set of possible values n. This is in keeping with the historical development of statistical noise, where an event is a possible outcome or outcomes.

An uncertainty measure is different from a Lebesgue measure [6] or a fuzzy measure [4, 5, 14, 15]. An uncertainty measure is a function $g : \mathcal{B} \rightarrow [0, 1]$ intended to measure the effects of noise. Three specific measures will be developed: the probability measure Pr, the possibility measure Π, and a necessity measure N_α.

In keeping with the terminology of probability, an element of $\mathcal{B}$ is called an *event*. The measure g is to be interpreted as a formalization of the ability of an event to occur.

The σ-algebra $\mathcal{B}$ is determined by the designer. $\mathcal{B}$ characterizes the ability of the designer to make statements about the NPS. The number of subsets within $\mathcal{B}$ is determined by the designer's ability to characterize the NPS.

That events in the NPS have the structure of a σ-algebra must be justified. Formally, this means the NPS has an associated collection of subsets $\mathcal{B}$ that satisfy, $\forall\ N_i, N_j \in \mathcal{B}$:

$$\begin{aligned} i) &\quad \overline{N_j} = \text{NPS} \backslash N_j \in \mathcal{B} \\ ii) &\quad \textstyle\bigcup_{j \in J} N_j \in \mathcal{B} \end{aligned} \tag{3.3}$$

where J is an index set. Thus, $i)$ states that for all events N_j in the set $\mathcal{B}$, not-an-event ($\overline{N_j}$) is equal to the Noise Parameter Space with the event (N_j) removed. It also states that if event N_j can occur, then $\overline{N_j}$ can also occur (or, stated differently: N_j can not occur). Also, $ii)$ states that the union of all events N_j (where j is in the index set J) are in the set $\mathcal{B}$. If index set J contains 1 and 2, and N_1 and N_2 are contained in $\mathcal{B}$ (which means that N_1 and N_2 can occur separately), then the union of N_1 and N_2 can occur. Stated differently: either N_1 or N_2, or both can occur. These are true for any sequence of events. DeMorgan's laws also hold on subsets:

$$\begin{aligned} iii) &\quad \overline{(N_i \cup N_j)} = \overline{(N_i)} \cap \overline{(N_j)} \\ iv) &\quad \overline{(N_i \cap N_j)} = \overline{(N_i)} \cup \overline{(N_j)}. \end{aligned} \tag{3.4}$$

Thus $iii)$ states that the complement (negation) of the union of N_i and N_j is the same as the intersection of not-N_i and not-N_j. Stated differently: if either N_i or N_j do not occur, then both N_i and N_j do not occur. Finally, $iv)$ states that the complement (negation) of the intersection of N_i and N_j is the same as the union of not-N_i and not-N_j. Stated differently: if neither N_i or N_j can occur together, then either N_i or N_j can not occur.

These assumptions are sufficient to show the collection $\mathcal{B}$ forms an algebra over the subsets [6]. Thus, events in $\mathcal{B}$ also satisfy:

$$\begin{aligned} ii) \quad & \bigcup_{j \in J} N_j \in \mathcal{B} \\ v) \quad & \bigcap_{j \in J} N_j \in \mathcal{B} \end{aligned} \tag{3.5}$$

where J is an index set. The algebraic structure of a noise space is not new to this work, it is historically well developed [6] (for probability).

Given an uncertain space (NPS,$\mathcal{B}$,g), the NPS is characterized. What is desired is to rate a design configuration, given these noise effects. To do so, a disjoint collection of subsets whose union is the entire NPS, or a *partition*, is needed. On each subset in the partition, the effects of the noise will be determined, and then each rating on each subset in the partition will be incorporated into an overall rating across the partition (across the NPS). Not just any partition is used, but the limit in refinement of any sequence of partitions within $\mathcal{B}$. Thus, the most accurate rating of the noise is used, given what the designer can state about the noise.

Probabilistic Uncertainty A particular uncertainty form that can be used to model the NPS occurs when the events are random. For example, inaccuracies in measurements and manufacturing are usually modeled as random. Such inaccuracies form what is now termed a *probability space*. The probability space will be denoted NPS, meaning all uncertainties in the NPS are now considered probabilistic, for this section.

Given the probability space, an uncertainty measure g is constructed, and denoted Pr. Pr measures the probability of an event occurring. A restriction of the Pr measure is that the probability of an event occurring and not occurring must equal the probability of the certainty event (assumed to be 1) under real addition. Further, the probability of either of two disjoint events occurring must equal the real additive probability of the two events. These restrictions are sufficient to derive an uncertainty measure Pr [2, 3, 4, 7, 18].

Thus, for events N_j, $N_k \in \mathcal{B}$ such that $N_j \cap N_k = \emptyset$, if g is restricted to obey:

$$\begin{aligned} g(N_j) + g(\text{NPS} \backslash N_j) &= 1 \\ g(N_j) + g(N_k) &= g(N_j \cup N_k) \end{aligned} \tag{3.6}$$

then g is a probability measure of classical probability theory [2, 7, 18].

If uncertainty information is included in design decision making, it is because the variational effects are to be minimized by proper selection of the nominal value of the imprecise variables. This is determined by weighting the preference of a design parameter set by its probability of occurring through the probabilistic uncertainty. This implies that given a probability space $(\text{NPS}, \mathcal{B}, Pr)$, the preferential performance of a point $d \in \text{DPS}$ is defined by

$$\mu(d) = \int_{NPS} \mu(d,n)\ dPr \tag{3.7}$$

where $\mu(d,n) = \mathcal{P}(\mu_1, \ldots, \mu_N)$. This integral is the standard Lebesgue integral from measure theory [6, 12]. Thus, $\mu(d)$ is the probabilistic expectation of $\mu(d,n)$ across the probability space with respect to the probability measure Pr. The design configuration d^* to use is the one which maximizes the expected performance of Equation 3.7 across the DPS.

If the NPS is a discrete NPS $\simeq \mathbb{Z}_q$, then each of the individual events $\{n_j\}$ forms a suitable partition, where each has an associated discrete $Pr(\{n_j\})$. The integral of Equation 3.7 then becomes a simple finite sum. If the NPS $\simeq \mathbb{R}$, then a partition of the form $[n, n + dn)$ might be used. This can usually be characterized by a probability density function $pdf(n)$. The integral of Equation 3.7 then becomes

$$\mu(d) = \int_{NPS|d} \mu(d,n)\ pdf(n|d)\ d(n|d) \tag{3.8}$$

a Riemann integral over the NPS. Since the distributions over n can, as before, vary with d, we here use the notation $n|d$. Of course the PPS, preferences, and density functions must all be Riemann integrable for this to hold.

This definition will produce a change in the "best" overall solution from the case without probabilistic uncertainty. Consider a one parameter design, with a preference μ and a pdf probability density as shown in Figure 3.1. The pdf is the same for all design parameter values. The maximal *expected* preference, *i.e.*, the maximum of the $E[\mu]$ curve (which is the result of applying Equation 3.7), not the μ maximum, is found. This set (d^*) is shown on the design parameter axis.

The resulting $\mu(d)$ determined from the integral will be the expected value of preference given the probabilistic uncertainty. One could also evaluate the higher moments of this relation to determine the standard deviation, skew, and kurtosis. Indeed, one could perform a Monte-Carlo simulation for each design parameter set $d \in \text{DPS}$ or, if possible, analytically determine the probabilistic uncertainty distribution of the $\mu(d)$. This additional information, however, is usually less important and is computationally expensive.

For designs with no probabilistic uncertainty, the parameters in the design model that are not design parameters are all crisp numbers. In such circumstances, Equation 3.7 reduces to Equation 3.1, since the probability density

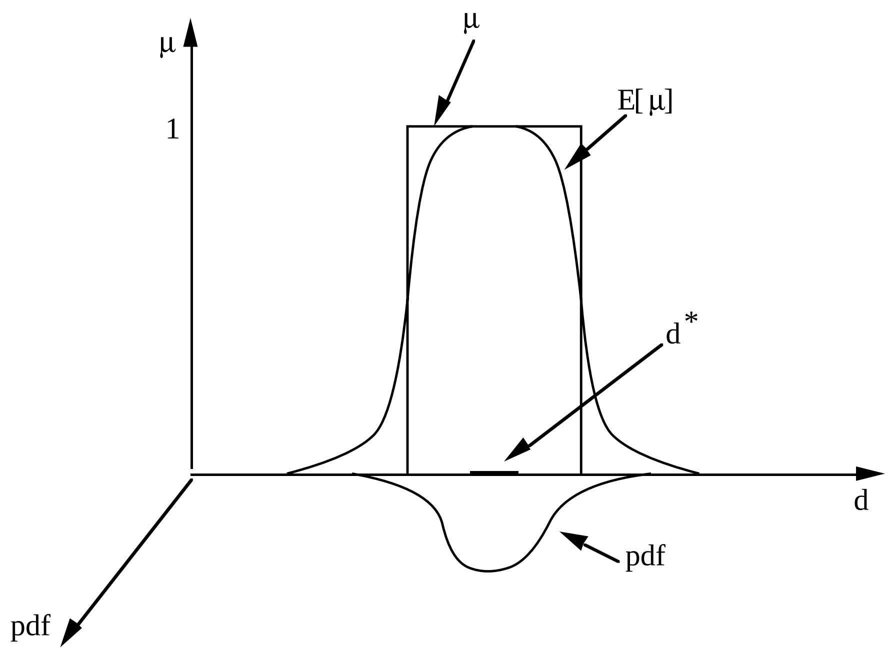

Figure 3.1 Parameter resolution with probabilistic uncertainty.

functions reduce to delta functions at the crisp values, which the integral of Equation 3.7 isolates to produce Equation 3.1.

The relation between this evaluation and Taguchi's method and experimental design in general can also be demonstrated. The solution to Taguchi's method has been shown to be an approximation to the solution which gives the highest quality (with a suitable definition of quality) [9]. The definition in Equation 3.7 and Taguchi's method are similar in this quality concept: Taguchi's method incorporates an experimental approximation to the integral in Equation 3.7 across probabilistic noise. The difference is that Taguchi's method is finding the mean of a single f (the S/N ratio), not preference over many design parameters and performance parameters.

The relationship this method and experimental design have is demonstrated by observing that the integral of Equation 3.7 can be approximated by experimental points in the noise space; *i.e.*, experimental design techniques can be used. These points can be chosen using a factorial method. Fractional factorial methods are possible and suggested. See [1, 9] for a discussion of factorial methods and methods for determining experimental points.

Possibilistic Uncertainty Possibility is the uncertainty in the limits of capacity within a formal model. Possibility can be used to represent parameters within a formal model that the designer does not have choice over, and that are not characterized by probability. Subjective choices of others (not the designer), for example, can be modeled with possibility. Thus, a possibilistic variable can have a range of values, but the range is limited by another person's choices.

Similarly to the previous form of probability, possibility will be formalized into what is now termed a *possibility space*. The possibility space will be denoted NPS, meaning all uncertainties in the NPS are now considered possibilistic, for this section.

Given the possibility space, an uncertainty measure g is constructed, and denoted Π which measures the possibility of an event occurring. For all events, either the event or not-the-event is possible. Also, if two events are possible, a choice must be made between them (but not by the designer). If two events can occur, it is assumed that some other person or agent will choose to maximize the performance (such as an adjustment or tuning during manufacture, testing, or operation). This means that when using either of two events, the performance will be the greater of the two, since this option would be chosen. Therefore, for possibilistic events $N_j, N_k \in \mathcal{B}$:

$$\begin{aligned} \max\{g(N_j), g(\text{NPS} \backslash N_j)\} &= 1 \\ g(N_j \cup N_k) &= \max\{g(N_j), g(N_k)\}. \end{aligned} \tag{3.9}$$

Equations 3.9 defines a possibility measure [5].

This formalism allows the measurement of the effects of noise on the performance, given any fixed design parameter arrangement d. Across the NPS, a disjoint collection of subsets $N \in \mathcal{B}$ whose union is the whole NPS is used (a partition of the NPS). The effect of each possible event N can be accounted for by determining the performance μ at a point $n \in N$, and then ensuring $\Pi(N)$ is within capacity at this evaluation point. Since the design is limited by the capacity Π, the design can be rated no better than the capacity Π. After this attenuation, the best possibility in the NPS can be used.

Thus, the integral of performance across the possibility space becomes the maximum possible performance, with performance attenuated to be possible to the degree specified by the possibility measure. Given a possibility space $(\text{NPS}, \mathcal{B}, \Pi)$, the preferential performance of a point $d \in \text{DPS}$ is

$$\mu(d) = \sup\{\min\{\mu(d,n), \Pi(N)\} \mid N \in \{N_j\} \text{ disjoint} \subset \mathcal{B}\} \tag{3.10}$$

where $\mu(d,n) = \mathcal{P}(\mu_1, \ldots, \mu_N)$, $n \in N$. This integral is a form of the Sugeno integral of possibility theory [14]. Note the *sup* is across the subsets of the partition $\{N_j\}$, and the limit as the partition becomes finer in $\mathcal{B}$ is used. Thus, $\mu(d)$ is the possibilistic expectation of $\mu(d,n)$ across the possibilistic noise space with respect to the possibility measure Π.

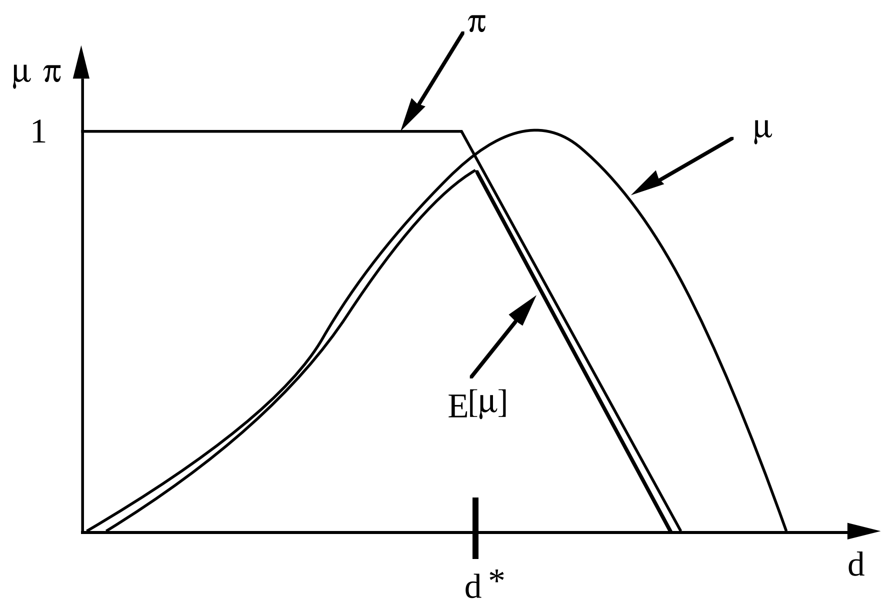

Figure 3.2 Parameter resolution with possibilistic uncertainty.

This definition resolves a different best solution. Consider a one parameter design, with a μ preference and a π possibility density as shown in Figure 3.2. The maximum expected preference, *i.e.*, the maximum of the $E[\mu]$ curve (using the possibilistic expectation of Equation 3.10), not the μ maximum, is found. This set is a point (d^*) as shown on the design parameter axis.

If all the points in the noise space are equally possible, ($\Pi(N) = 1 \; \forall \; N \in \mathcal{B}$), then the development reduces to a simple max of $\mu(d, n)$ across the noise and design space, *i.e.*, this reduces to finding the max of $\mu(d)$ across the design space, as shown in Equation 3.1.

3. Necessary Parameters

So far this presentation has assumed that all the parameters can take on any value within their distribution ranges. Alternatively, the design may need to satisfy *every* value of a noise parameter to the degree specified by the distribution. Such parameters are termed *necessary parameters* [19].

A design's noise parameters can be grouped into those that must be satisfied across their entire range of variation, and those which any single value in the allowable range can be used. This distinction corresponds to whether a parameter should be modeled as a necessary parameter.

Two forms of necessary parameters are recognized: probabilistic necessity and possibilistic necessity. Probabilistic necessity arises when a parameter varies probabilistically and the designer wishes to ensure the design will function for the entire range of variation. Possibilistic necessity arises in a similar situation, but when the parameter varies possibilistically.

When using necessity, the degree of satisfaction desired for a necessary parameter must be specified, and shall be denoted α. α is therefore a number in $[0, 1]$, it is not a distribution, and reflects the domain percentage, as measured by the underlying uncertainty measure (either Pr or Π), that the designer feels must be ensured. α will be termed the *confidence factor*. If $\alpha = 0$, then only the most likely value is considered. If $\alpha = 1$, then all values in the NPS must be satisfied. For some design problems, α might need to be expressed absolutely, *i.e.*, $\alpha = 0.999999$ to satisfy designs to six standard deviations. For others, α might be made a function of the preference μ, so when the achievable preference is high, the necessity range is increased (α is increased); when the achievable preference is low, the necessity range is decreased (α is decreased), easing the degree of difficulty in satisfying the design.

Thus, there is a requirement of determining the domain (NPS) percentage required to be satisfied. A subset $\mathcal{N}_\alpha \in \mathcal{B}$, called the *necessary set*, is defined as the subset of the NPS that is desired to be satisfied.

Given the necessary subset $\mathcal{N}_\alpha$, any other subset N in $\mathcal{B}$ can be identified as necessary. If N lies within $\mathcal{N}_\alpha$, then the subset N is necessary, otherwise it is not. Thus, a necessity measure of each subset N can be constructed given the necessary subset $\mathcal{N}_\alpha$. This measure can be defined by

$$\mathrm{N}_\alpha(N) = \begin{cases} 0 & N \subseteq \mathcal{N}_\alpha \\ 1 & N \not\subseteq \mathcal{N}_\alpha. \end{cases} \tag{3.11}$$

Any measure g satisfying Equation 3.11 will be called a *necessity measure*, and is denoted by N_α. Thus, only if all points within a set N are necessary does the set N become necessary.

The necessity measure is a $\{0, 1\}$ placed measure. The designer uncertainty will be incorporated into $\alpha \in [0, 1]$, to determine the extent of $\mathcal{N}_\alpha$. Thus the uncertainty is incorporated into specification of $\mathcal{N}_\alpha$ (the necessary set), not in N_α (the measure of a set in $\mathcal{B}$).

An uncertainty integral can be constructed with the necessity measure. The integral of performance becomes the worst case performance across the necessity space, as measured by the necessity measure. Given a necessity space $(\text{NPS}, \mathcal{B}, \mathrm{N}_\alpha)$, the preferential performance of a point $d \in \text{DPS}$ is defined by

$$\mu_\alpha(d) = \inf \left\{ \max\{\mu(d, n), \mathrm{N}_\alpha(N)\} \mid N \in \{N_j\} \text{ disjoint} \in \mathcal{B} \right\} \tag{3.12}$$

where $\mu(d, n) = \mathcal{P}(\mu_1, \ldots, \mu_N)$, $n \in N$. Again, the limit as the partition becomes finer in $\mathcal{B}$ is used. Thus, $\mu_\alpha(d)$ is the necessary expectation of $\mu(d, n)$ across the necessary space with respect to the necessary measure N_α.

The problem is thus well formed, provided the set $\mathcal{N}_\alpha$ can be identified. How much of the NPS should be ensured? For an NPS $\simeq \mathbb{R}^q$ independent, this has typically been done with *confidence intervals* [16]. This formalism, however, assumes that the underlying uncertainty measure (Pr or Π) is constructed from a density function (pdf or π), which requires an ordering on the NPS. In any case, the portion of the uncertainty space to be satisfied is identified as the *confidence factor*: α.

3.1 Probabilistic Necessary Parameters

When the underlying uncertainty is probabilistic, the density function that can be used to partition an ordered NPS is the probability density function, pdf : NPS $\rightarrow \mathbb{R}^+ \cup \{0\}$. The pdf can be used to define the necessary subset of the NPS, for any confidence factor α:

$$\mathcal{N}_\alpha = \{n \in \text{NPS} \mid pdf(n) \geq \Theta\} \tag{3.13}$$

where

$$\Theta = \inf \left\{ \theta \;\middle|\; pdf(n) \geq \theta \text{ and } \int_{NPS} \chi_{\{pdf(n) \geq \theta\}}(n) \, dPr \leq \alpha \right\}. \tag{3.14}$$

Thus, Θ is the lowest level pdf value with α equal to the Pr of all n whose $pdf \geq \Theta$. This forms the set of elements n whose total $Pr = \alpha$, and whose n all have $pdf \geq \Theta$.

For example, in the case of NPS $\simeq \mathbb{R}$, and pdf as the normal distribution, $\mathcal{N}_\alpha$ becomes a class interval:

$$\mathcal{N}_\alpha = [E - r, E + r]$$

where E is the expected value, and r is a radial distance from the expected value such that $Pr([E - r, E + r]) = \alpha$.

The most preferred design parameter set and the maximal preference in light of probabilistic necessary parameters n then becomes:

$$d^* : \mu_\alpha(d^*) = \sup \{\inf \{\mu(d, n) \mid n \in \mathcal{N}_\alpha\} \mid d \in DPS\}. \tag{3.15}$$

To see this definition's meaning, consider a one variable design with a probabilistic necessary distribution, as shown in Figure 3.3. Each value along the design parameter axis is uncertain because of the normal probabilistic variation as shown. Therefore the preference curve μ must be reduced at each point

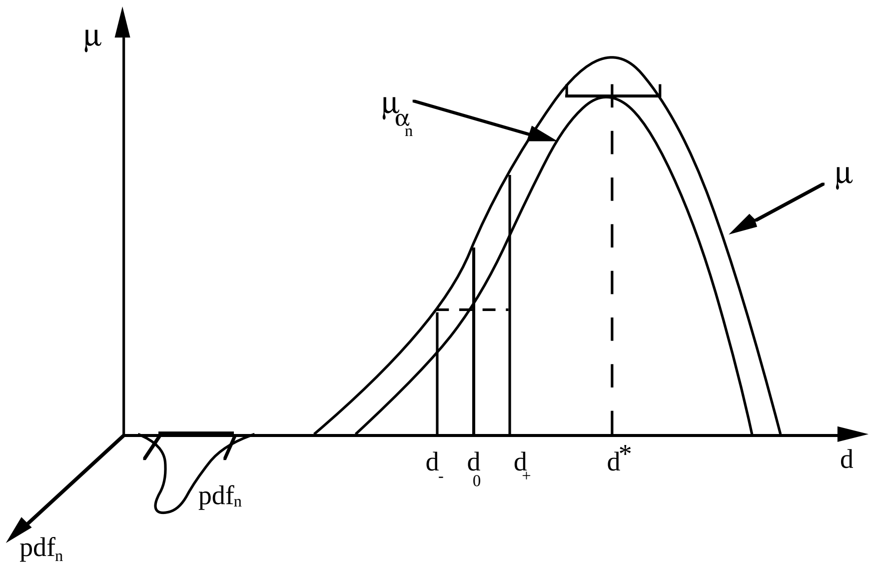

Figure 3.3 Probabilistic necessary parameter preference resolution.

to the lowest preference it can take on, given a probabilistic variation, and a degree. For example, as shown in Figure 3.3, at a design parameter value d_0 a 95% confidence interval ($\alpha = 0.95$) can produce variations in the range from d_- to d_+. The lowest preference μ in that range $[d_-, d_+]$ becomes the μ_α for d_0. This is repeated for all design parameter points to obtain the μ_α curve. The maximum of this μ_α curve is the "best" solution, that is, the most preferred design parameters subject to the necessary probabilistic distribution.

3.2 Possibilistic Necessary Parameter

When the underlying uncertainty is possibilistic, the density function which can be used to partition the NPS is the possibility density function $\pi : \text{NPS} \rightarrow [0, 1]$. π can be used to define the necessary subset of the NPS, for any confidence factor α:

$$\mathcal{N}_\alpha = \{n \in \text{NPS} \mid \pi(n) \geq 1 - \alpha\}. \tag{3.16}$$

The most preferred design parameter set and the maximal preference in light of possibilistic necessary parameters n then becomes:

$$d^* : \mu_\alpha(d^*) = \sup \{\inf \{\mu(d, n) \mid \pi(n|d) \geq 1 - \alpha, n \in \text{NPS}\} \mid d \in \text{DPS}\}. \tag{3.17}$$

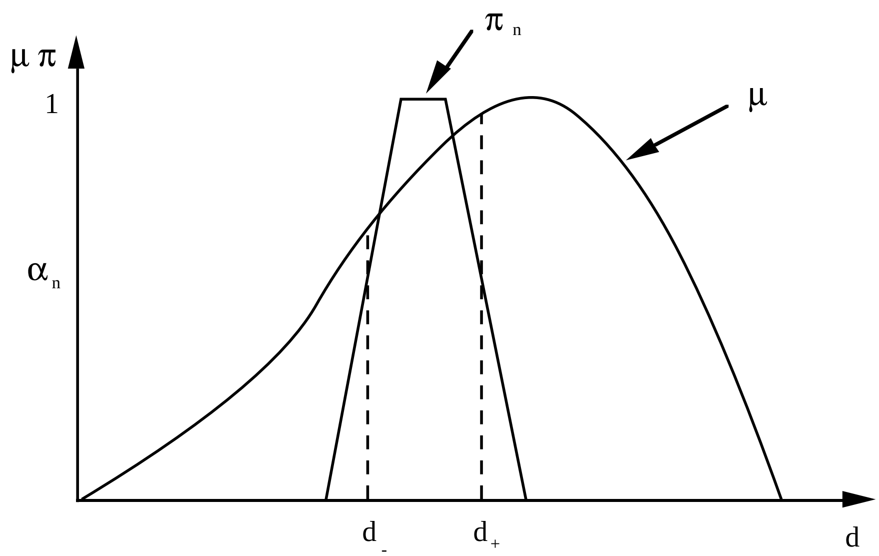

Figure 3.4 Possibilistic necessary parameter preference resolution.

This definition is illustrated in Figure 3.4. Here the π denotes the necessary region, and n is considered as the design parameter ($n = d$). Therefore the solution is the entire support of the π distribution, and the object is to rank the degree of preference for the range. Therefore the preference μ_α must be the lowest preference within the support of π. For example, as graphed in Figure 3.4, at a degree of necessity α, the domain of necessity is from d_- to d_+. The lowest preference μ in that range $[d_-, d_+]$ becomes the μ_α for the necessary range.

4. Hybrid Uncertainty

For problems with multiple uncertainty forms, Equations 3.7 through 3.17 must be combined. Such a combination is possible, but requires making explicit the *precedence relation* among the parameters. This is discussed in [9, 10] for Taguchi's method and optimization.

As an example, consider the design of a uni-directional accelerometer, which indicates accelerations above a threshold with a switch closure. It can be modeled as a simple mass spring system, as shown in Figure 3.5. Under specified accelerations, the accelerometer mass must contact a switch within specified time durations. Suppose, however, that the spring is a thin metal sheet manufactured by a stamping procedure. The inaccuracies introduced by the stamping manifest themselves as a variation in the value for k, the spring constant. This uncertainty occurs randomly. Hence due to the manufacturing process, it

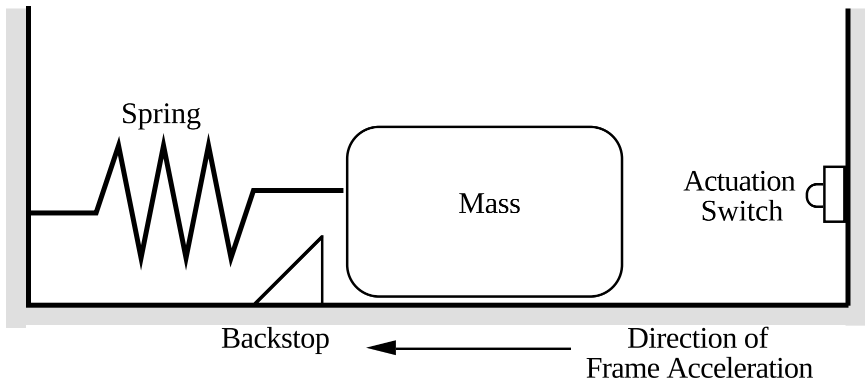

Figure 3.5 Example: accelerometer.

is difficult to set precise actuation times (time for the mass to move into contact with the actuation switch). The design has, however, a method to overcome these manufacturing errors in the spring. Specifically, during manufacturing, the backstop of the mass can be adjusted to compensate for variations in k. This backstop positioning distance is a tuning parameter of the design. During manufacture, the spring constant of every accelerometer is measured, and the backstop of each accelerometer is positioned accordingly to meet the specified actuation times.

Such parameters are denoted *tuning parameters*, and are introduced in [10]. They are not an artifact of the imprecision formulation, they exist in any formulation. They must be accounted for, however, when selecting the design parameters.

Such relations between design, confounding, and tuning parameters are readily modeled using imprecision. A tuning parameter's range of possibility forms a possibilistic uncertainty. Therefore one can combine Equations 3.7 through 3.17, but care must be taken that the equations are combined in their proper order: design parameters on the outside, and tuning parameters on the inside (relative to the confounding noise parameters).

Thus, with multiple forms of uncertainty, the d^* are chosen as

$$\mu(d^*) = \sup \left\{ \int_{Pr(\delta d)} \sup \left\{ \mu(d, \delta d, t) \mid t \in \text{TPS} \right\} \times dPr(\delta d) \,\middle|\, d \in \text{DPS} \right\} \tag{3.18}$$

where δd are the manufacturing errors and t are the tuning parameters. The maximization of the tuning parameter selection occurs inside the integral across the manufacturing errors, and the maximization of the design parameter selection occurs outside the integral across the manufacturing errors.

It is not always true that possibilistic uncertainty has its evaluation inside the probabilistic integral. If there had been possibilistic uncertainty associated with the design parameters d to limit the designer's choice, then this combination occurs outside the manufacturing error integral. Similarly, if the designer had particular preferences for tuning parameter values, then this combination occurs inside the integral. The precedence relation among the variables is determined by the variable type (design, noise, and tuning parameters) not on the uncertainty forms associated with each parameter (imprecision, probability, and possibility).

For a general design problem, the evaluation order of the maximizations, minimizations, and integrals will depend on the precedence relation among the variables. This is not an artifact of the imprecision formulation. The same problem occurs with any other formulation (such as probabilistic optimization or an extended form of Taguchi's method); the reader is referred to [10] for demonstration of the precedence relation ordering in these other formulations. Imprecision simply sets the metric across the space to be preference (μ) rather than, for example, a single performance parameter expression.

Finally, the tuning parameter's value is a possibilistic uncertainty from the design engineer's perspective. It has a range of possible values and the value that should be used cannot be set by the design engineer, since it will depend on the manufacturing errors. But the expected value of the probabilistic manufacturing error can be determined. Since the possibilistic tuning parameter's values depend on the probabilistic manufacturing error, then from the design engineer's viewpoint (pre-manufacturing) the tuning parameter expected value can also be found. That is, the possibilistic tuning parameter will adjust its value based on the probabilistic manufacturing error. Hence, there will be, from the design engineer's viewpoint, a probabilistic distribution for the tuning parameter as well, even though it has no probability aspect associated with it at all. Taking a view of the tuning parameter before the noise occurs, it can be said to have a distribution. But, inherent in the parameter itself (*i.e.*, from the manufacturing engineer's perspective who must actually set the variable's value after the noise has occurred), the parameter has absolutely nothing to do with probability.

5. Example

The example presented herein considers the design of a pressurized air tank, and is the same problem as presented in Papalambros and Wilde [11], page 217. The reader is referred to reference [11] to see the restrictions applied to the problem to permit it to be solved using crisp constraints and an optimization methodology. The same problem is considered in a previous paper with no uncertainty, only imprecision and preferences [8]. The example is simple

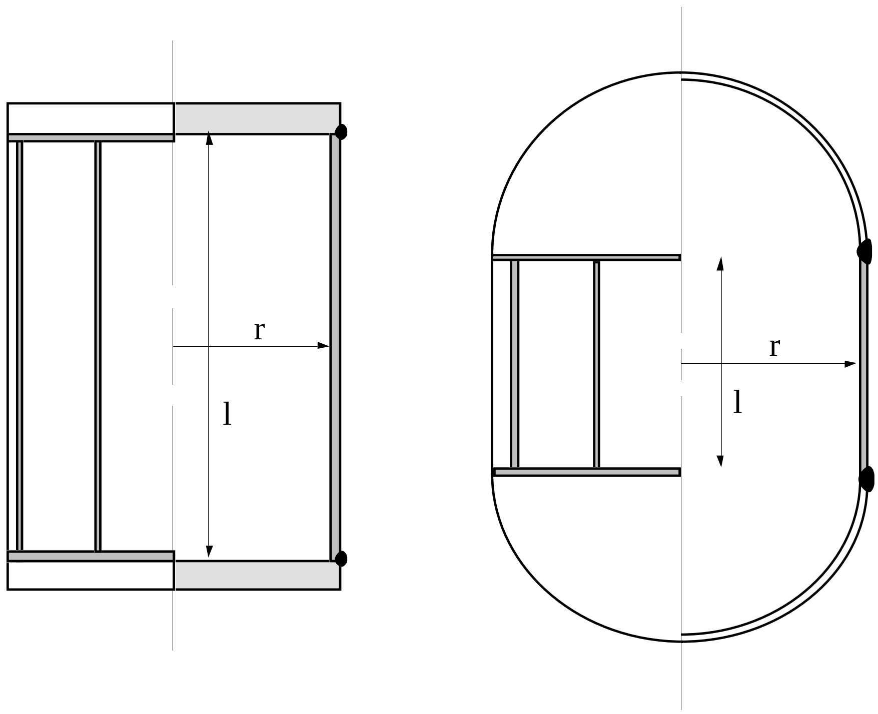

Figure 3.6 Hemispherical and flat head air tank designs.

but was chosen for that reason, and also the ability of its preferences to be represented on a two-dimensional plane for a visual interpretation.

The design problem is to determine length and radius values in an air tank with two different choices of head configuration: flat or hemispherical. See Figure 3.6.

There are four performance parameters in the design. The first is the metal volume m:

$$m = 2\pi K_s r^2 l + 2\pi C_h K_h r^3 + \pi K_s^2 r^2 l \tag{3.19}$$

This parameter is proportional to the cost, and the preference ranks of m are set because of this concern. Another is the tank capacity v:

$$v = \pi r^2 l + \pi K_v r^3 \tag{3.20}$$

This parameter is a measure of the attainable performance objective of the tank: to hold air, and the desired level of this performance ranks the preference for values. Another performance parameter is an overall height restriction L_0, which is imprecise:

$$l + 2(K_l + K_h) r \leq L_0 \tag{3.21}$$

Finally, there is an overall radius restriction R_0, which is also imprecise:

$$(K_s + 1)r \leq R_0 \tag{3.22}$$

The last two performance parameters have their preference ranks set as a result of spatial constraints.

The coefficients K are from the ASME code for unfired pressure vessels. S is the maximal allowed stress, P is the atmospheric pressure, E is the joint efficiency, and C_h is the head volume coefficient.

$$K_h = \begin{cases} 2\sqrt{CP/S} & \text{flat} \\ \frac{P}{2SE-.6P} & \text{hemi} \end{cases} \tag{3.23}$$

$$K_l = \begin{cases} 0 & \text{flat} \\ 4/3 & \text{hemi} \end{cases} \tag{3.24}$$

$$K_s = \frac{P}{2SE - .6P} \tag{3.25}$$

$$K_v = \begin{cases} 0 & \text{flat} \\ 1 & \text{hemi} \end{cases} \tag{3.26}$$

This example's design space is spanned by 2 design parameters l and r. The preferences for values of these design parameters and the four performance parameters are shown in Figures 3.7 through 3.12.

The problem, however, is confounded by noises. There are manufacturing errors on l and r that limit how well one can specify their values. We assume, for this example, that this error is Gaussian, however, noise with any distribution can be incorporated in the same manner. Error is also introduced by the supplied material variability. This error is manifested in the allowable stress S, which varies (in this example) with a beta distribution. The effects of these errors are desired to be minimized.

Finally, there is error introduced in the variability of the welds made. This error is manifested in the joint efficiency E, which varies (in this example) with a beta distribution. The effects of these errors must be reduced such that even the least efficient weld, within the chosen tolerance, will not fail, since failure in a weld represents a safety concern. Therefore, this error is modeled as a necessary probabilistic uncertainty.

The other unknown in the problem is the applied pressure P, which can vary with use. This is represented as a range of possibilistic necessity from -15 and 120 psi, since this design must satisfy all these pressures.

These distributions are shown in Figures 3.13 through 3.16. The necessary parameters used a value of $\alpha = 0.9$, or the variables were satisfied 90 percent of the time. These delimiters are also shown in the figure.

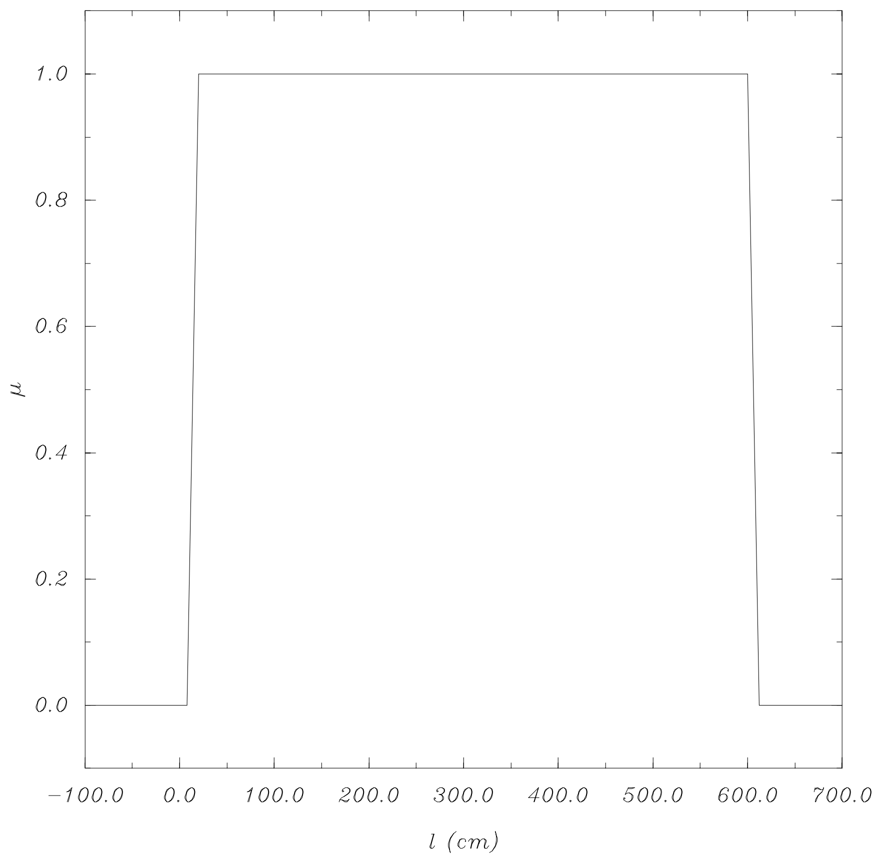

Figure 3.7 Length l preference.

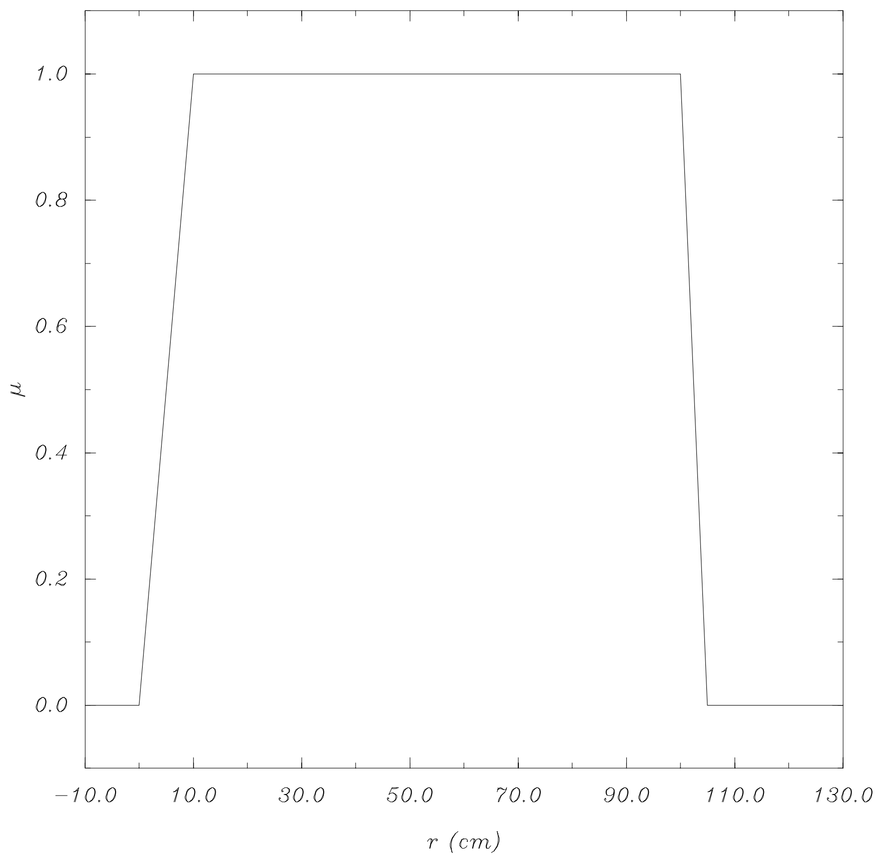

Figure 3.8 Radius r preference.

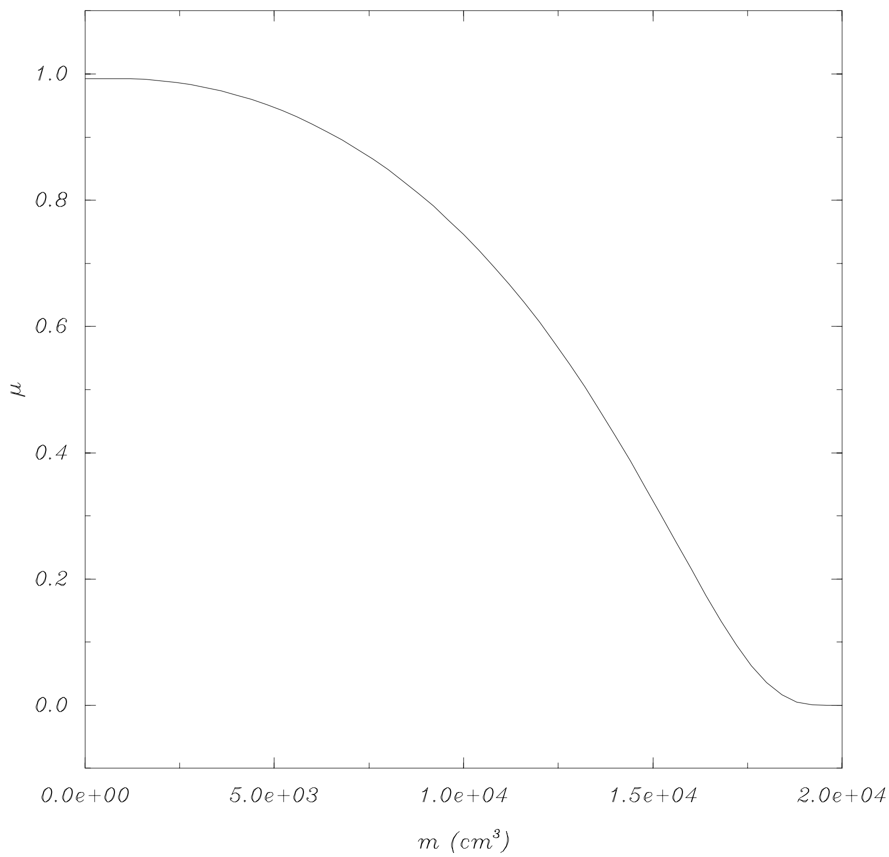

Figure 3.9 Metal volume m preference.

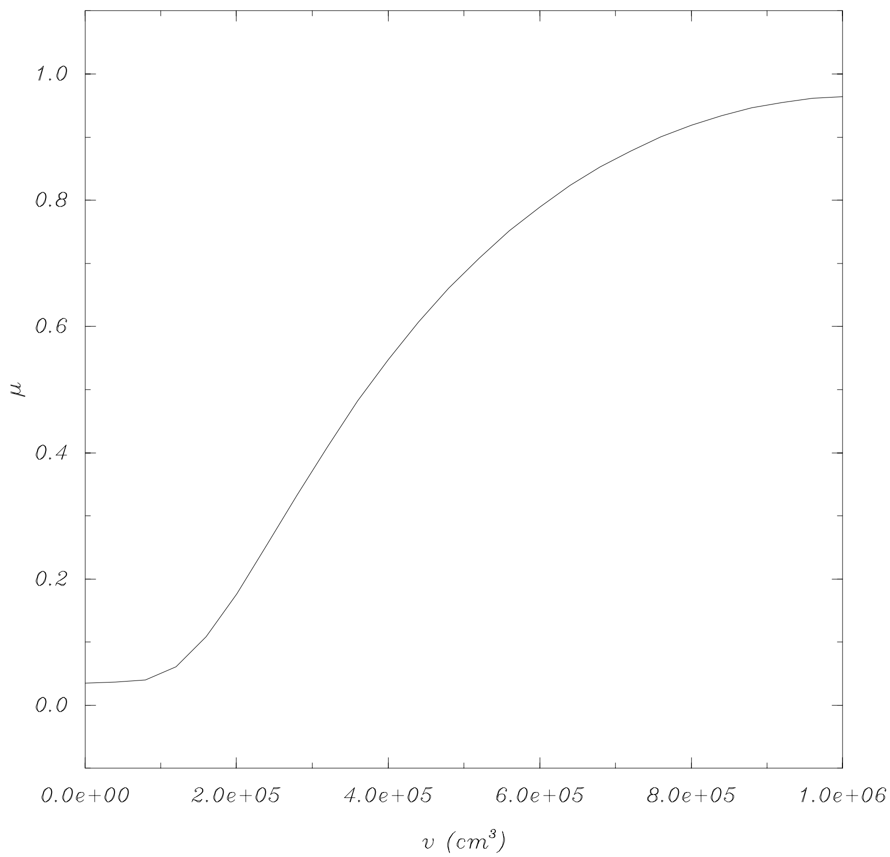

Figure 3.10 Capacity v preference.

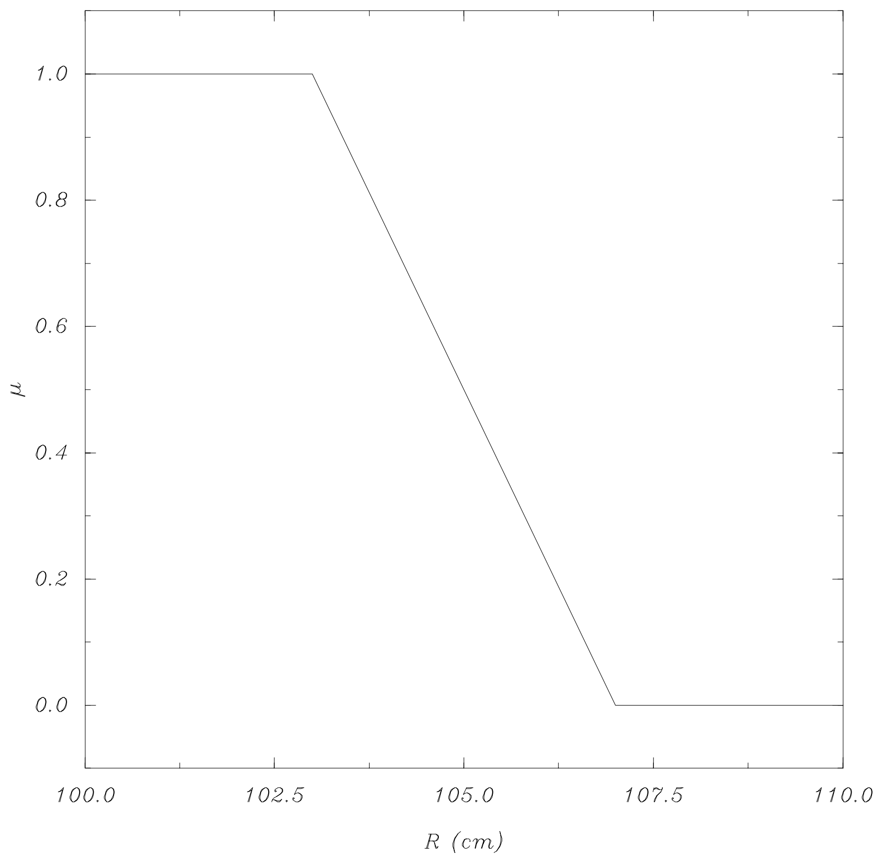

Figure 3.11 Outer radius R_0 preference.

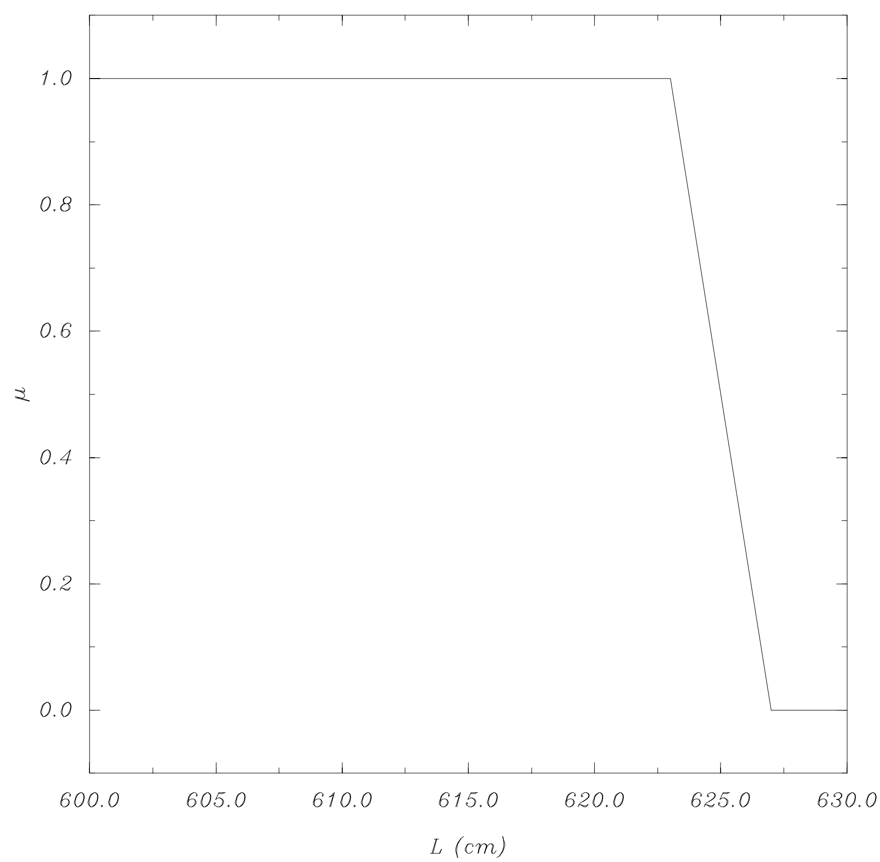

Figure 3.12 Outer length L_0 preference.

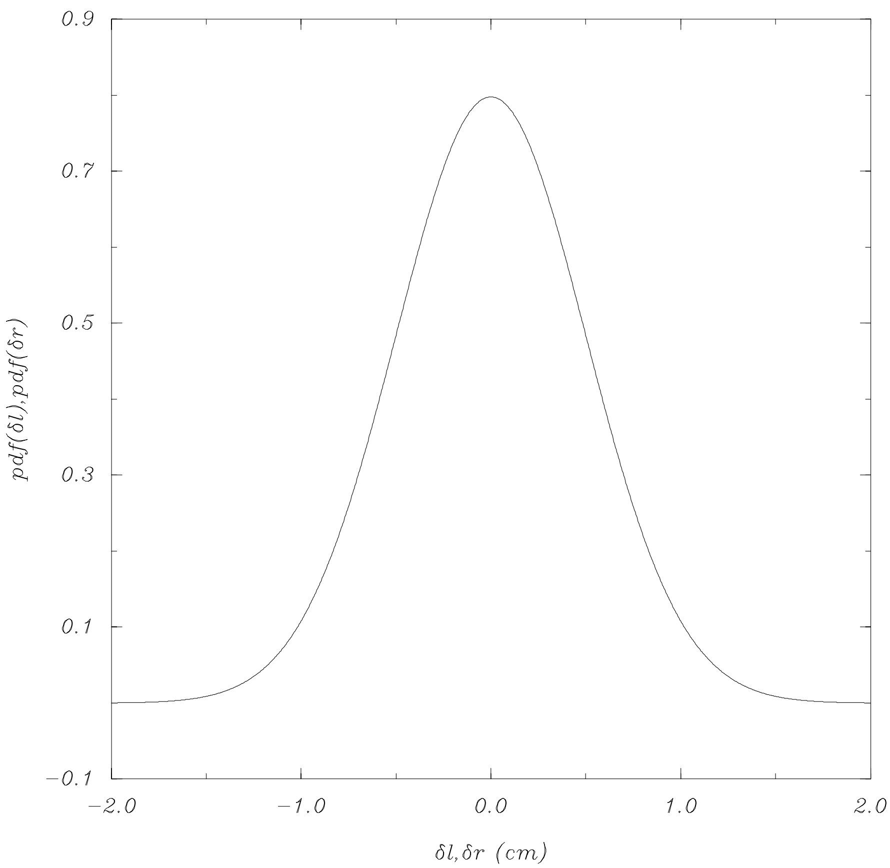

Figure 3.13 Length l and radius r uncertainty distribution

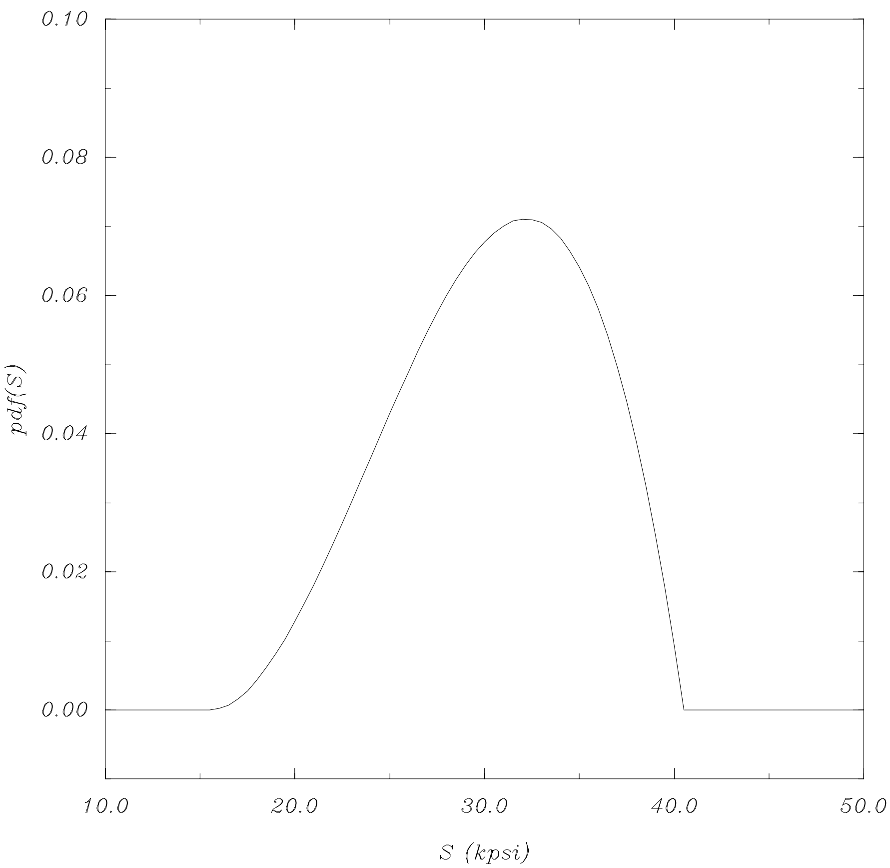

Figure 3.14 Allowable stress S uncertainty distribution

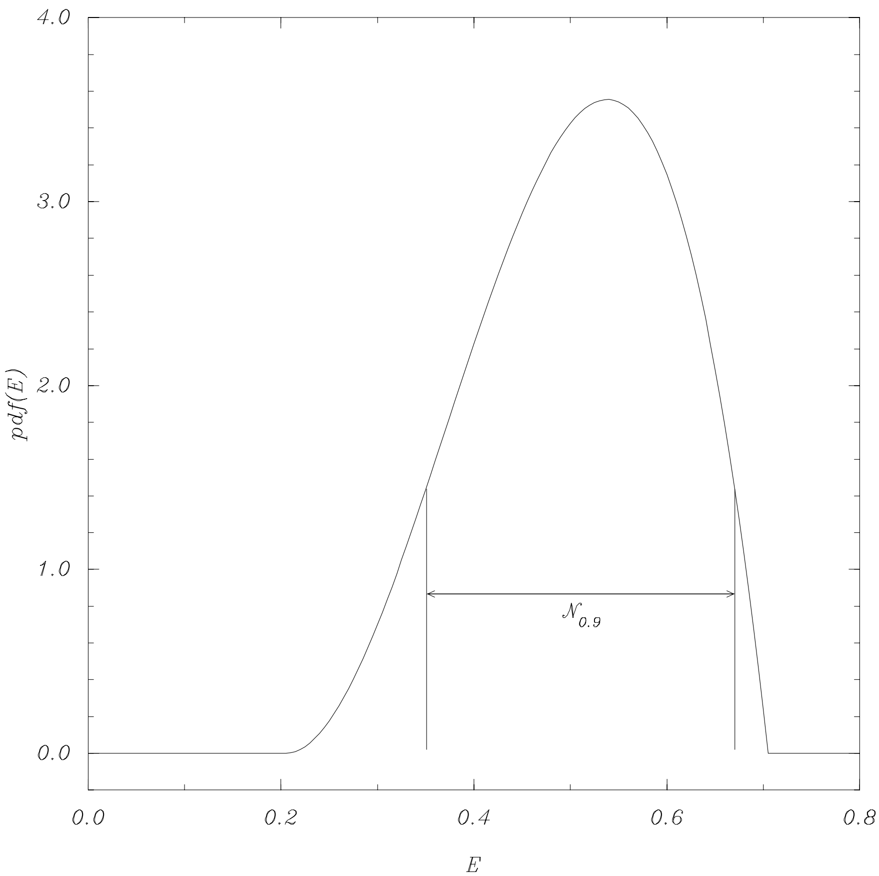

Figure 3.15 Joint efficiency E uncertainty distribution

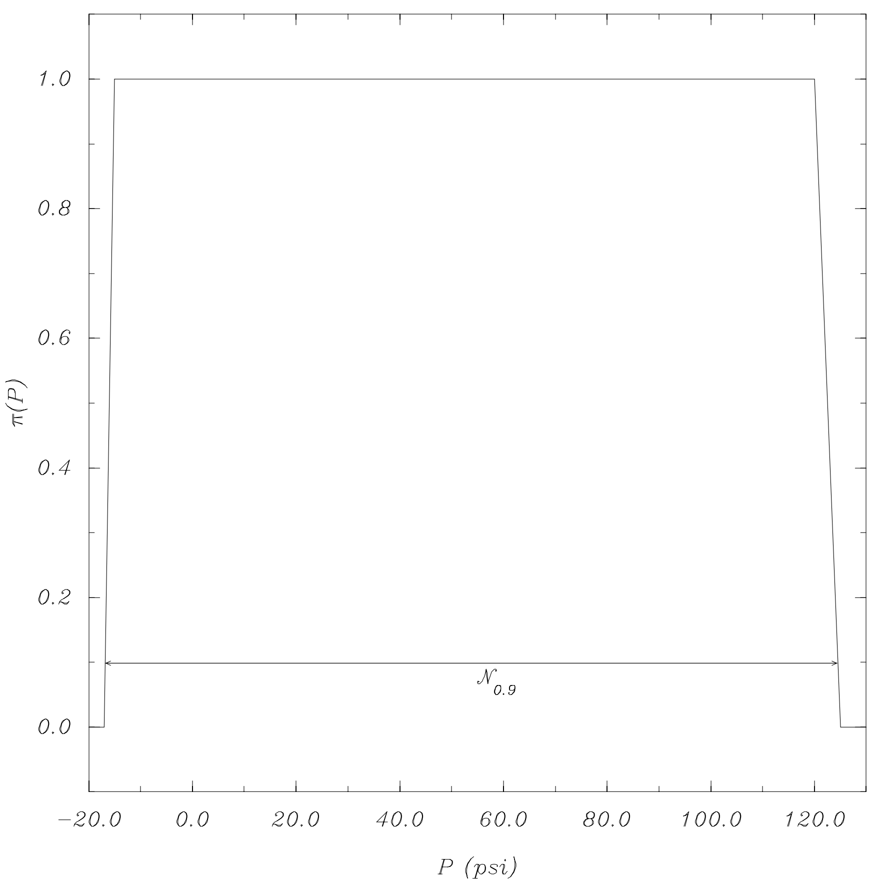

Figure 3.16 Applied Pressure P possibility distribution

The problem, then, is to find the values for l and r that maximize overall preference. It is yet to be determined how to evaluate this objective – *i.e.*, a strategy must be set [8]. In this particular example the non-compensating design strategy will be used. This means that among the multiple goals of the design, the worst performing goal will be improved, if any improvement can be made at all, by changing values of the design parameters l and r.

For a non-compensating design strategy, the problem to be solved is to find l^*, r^*, where:

$$\mu(l^*, r^*) = \sup\left\{\inf\left\{\int_S \int_{\delta r} \int_{\delta l} \min\{\mu_l, \mu_r, \mu_v, \mu_m, \mu_{L_0}, \mu_{R_0}\}\; pdf(S)\, pdf(\delta r)\, pdf(\delta l)\;\; dS\; d(\delta l)\; d(\delta r) \;\middle|\; (P, E) \in \mathcal{N}_{0.9}\right\} \;\middle|\; (l, r) \in \mathbb{R}^2\right\} \tag{3.27}$$

This will determine the l^* and r^* that maximizes the poorest design aspect's preference, yet considers the confounding probabilistic noise effects, and satisfies the necessary parameters 90 percent of the time.

The design space is shown in Figure 3.17 for the flat head design and in Figure 3.18 for the hemispherical head design. The peak preference point represents the design parameter values to use: those with maximum expected preference, given the designer specified preference curves, necessary distributions, and the noise distributions. The results are different from the case when no noise or necessity was considered. When only preference information is considered (the expected values are considered for the noise parameters), the resulting l and r values are chosen directly on the imprecise constraint boundaries, as shown in a previous paper [8]. The consideration of noise moves the chosen parameter values from the imprecise preference curves to more robust values, as shown in Figure 3.17.

This differs from the results of the various problem formulations presented in Papalambros and Wilde [11]. For example, the non-linear programming formulation solves the problem by minimizing the metal volume with the rest of the goals as crisp constraints. The preference formulation allows the constraints to be elastic, so the final design parameter values determined are different than if crisp constraints had been used. If the example had selected step functions for preference curves on the constraint performance parameters and no noise was considered, the results would reduce to a non-linear programming solution. The addition of noise to the problem with step function constraints would reduce the problem to a probabilistic optimization formulation, as discussed in Siddall [13].

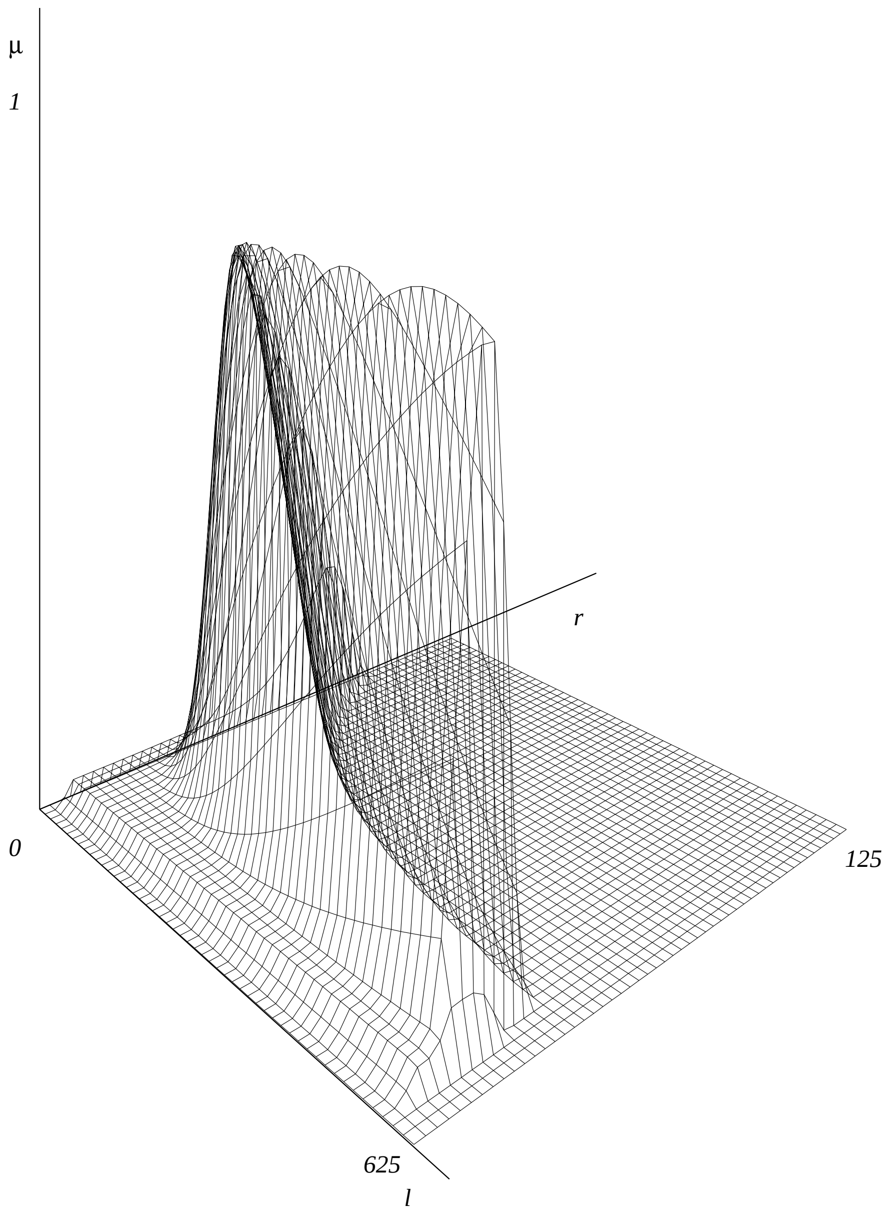

Figure 3.17 The expected preference across the design space (r,l) for the Flat head design.

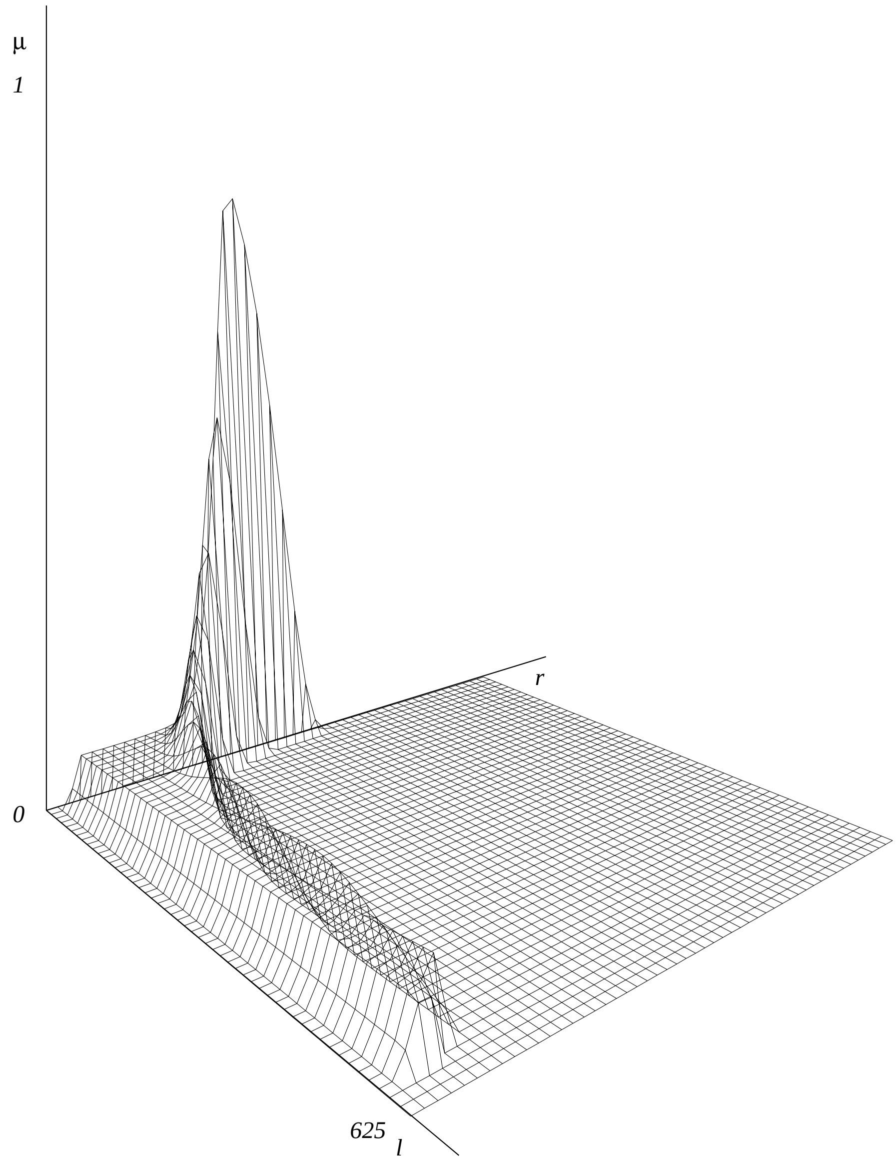

Figure 3.18 The expected preference across the design space (r,l) for the Hemispherical head design.

6. Conclusion

Imprecise preference functions have been used to solve design problems with multiple, incommensurate goals. A global metric is constructed across the design space from the preference rankings of the parameters. The result of applying the design's global metric has a simple interpretation as the overall preference for the design.

The imprecision methodology of resolving design parameter values is extended to include different confounding noise forms. Probabilistic noise requires the use of the probabilistic expectation process. This allows for the determination of the design parameter values that produce the highest overall quality. Possibilistic uncertainties, on the other hand, require the use of the possibilistic mathematics to confine the design to select the best value among only those that are possible.

The formulation is also extended to include necessary requirement forms. Necessary parameters can be either probabilistic or possibilistic in nature, and the parameter's necessity range is determined accordingly. Non-linear formulations are possible that tie α to μ. Such a formulation will permit trade-offs between preference and degree of necessity.

Design problems that have different variable types (design parameters, noise parameters, and tuning parameters) can now be solved. This work allows the designer to determine the "best" design parameter set to use, given uncertain design specifications and uncertainty in manufacturing processes.

Acknowledgments

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