Membership Induced on Manifolds by Vector Fields and Flows*

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Abstract

In many engineering and scientific applications, the need arises to perform differential calculations on manifolds. If such calculations involve imprecise data, methods for computing with such data must be determined. Definitions are given for memberships on manifolds induced by differentiation along vector fields, and by flows of vector fields with uncertain parameters. Examples are presented which illustrate the concepts.

Introduction

In many scientific and engineering applications, the need arises to perform calculations on manifolds [1, 2]. Many dynamical systems of science and engineering are most naturally framed and solved in such an environment. Revolute joints in mechanical systems, for

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example, naturally lead one to make calculations on tori. Performing calculations using simple real variables is not sufficient.

Given that calculations representing such systems must involve manifold calculations, it might be desirable to as well consider imprecision in the variables of the calculations. For example, when a mechanical designer is faced with designing a robotic manipulator, initially the precise link lengths or other variables to choose may not be precisely known. The designer may only be able to state subjectively what the values are approximately. Yet, given even these approximate specifications, the designer must calculate performance to be able to choose an actual configuration to use.

A natural approach to calculating the approximate performance given the approximate configurations would be to invoke the mathematics of fuzzy sets [3, 17], since the approximate nature of the problem is due to the imprecision in knowledge about which configuration will be used. Examples of such an approach in engineering can be found in [12, 13, 14, 15]. In attempting to do such calculations over manifolds with associated maps, however, one finds the existing literature lacking in definitions on how to do so.

Dubois and Prade [4] consider differentiation and integration of fuzzy numbers; that is, of variables whose domain is the simple manifold $\mathbb{R}$. Seikkala [11], Kaleva [6], Goetschel and Voxman [5], and Puri and Ralescu [10] perform the operations of differentiation and integration on a Banach space of fuzzy sets satisfying certain hypotheses. This paper, alternatively, considers the membership induced through mappings on the manifold itself. In some early work, Negoiţă and Ralescu [8] define induced membership for iterating a set map. Such work can be seen as a generalization of the o.d.e. work in this paper, but the generality to arbitrary sets limits its practicality.

In this paper some of the basic notions of fuzzy sets are developed for cases when the set has the structure of a differentiable manifold. Specifically, the membership induced by the Lie derivative of a function along a vector field, and the membership induced by the flow of a vector field, possibly depending on imprecise parameters, is defined. Thus the results of this paper are the definitions themselves, and their applications.

In Section 1 the basic notion of induced membership is discussed and some examples are given. In Section 2 the induced membership is defined for the derivative of a function along a given vector field. The usual notion of differentiation of functions on $\mathbb{R}^n$ with respect to the natural coordinates can be thought of as a specific example of this construction. Illustrative examples are given. Section 3 presents ideas of how membership may be propagated along the integral curves of a vector field. Two basic mechanisms which give rise to induced membership are examined. The first case is that where an initial membership on the manifold evolves as a function of time along the integral curves of the vector field. In the second case, the vector field is allowed to depend on parameters which have some imprecision associated with them. Both of these cases are examples of a more general construction which is a combination of both problems. Examples are presented for each case. Finally, Section 4 discusses an industrial example, the design of a mechanical accelerometer switch.

1 Induced Membership

The discussion in this section, and in the following sections, will be presented in the context of differentiable manifolds (we will always work in the C^∞ category). However, it should be clear that the concepts in this section are valid on the level of sets.

Let M be a smooth manifold. A *membership function*, or simply a *membership*, on M is a map $\mu : M \to [0,1]$. No smoothness requirements are placed on μ. Indeed, in the examples presented, none of the membership functions are smooth, and some of them are not even continuous.

In the particular case of $M = \mathbb{R}^N$, each $x \in \mathbb{R}^N$ can be thought of as a vector $x = (x^1, \ldots, x^N)$, where perhaps there are fuzzy numbers on each coordinate with membership functions $\mu_i(x^i)$, in which case a membership function for $x \in \mathbb{R}^N$ could be constructed by $\mu(x) = \min\{\mu_1(x^1), \ldots, \mu_N(x^N)\}$. But for general engineering and scientific purposes, the fuzzy number concept is insufficient: there is no general way to define fuzzy numbers on manifolds.

If N is another manifold and if $f : M \to N$ is a smooth map, we have the following definition.

1.1 Definition: The *membership induced by f* is the membership on N given by

$$\begin{aligned} \mu_f \quad : \quad & N \to [0,1] \\ & n \mapsto \sup\{\mu(x) \mid x \in M, \quad f(x) = n\} \end{aligned}$$

If $f^{-1}(n) = \emptyset$, then take $\mu_f(n) = 0$.

In many problems, $N = \mathbb{R}$, so that memberships are induced on the values of a real-valued function. Observe that this definition is nothing more than a restatement of the standard notion of induced membership [3, 17], as originally presented by Zadeh [16], to manifolds.

The examples which will be presented in this paper are meant to be illustrative. Given an application, it should be straightforward, in principle, to proceed using the methods outlined here.

1.2 Example: Let $M = \mathbb{R}$ be equipped with the membership function whose graph is shown in Figure 1. This example is thus illustrative of a fuzzy number. Define $f_1 : M \to \mathbb{R}$ by $f_1(x) = x^3 - x$. Figure 2 shows the membership induced on $\mathbb{R}$ by f_1. This illustrates the membership on values of the function f_1 given the membership on x as shown in Figure 1. This example will be returned to in Section 2.

1.3 Example: Let $M = \mathrm{S}^2 = \{\vec{x} \in \mathbb{R}^3 \mid \quad ||\vec{x}|| = 1\}$. If (x, y, z) are the standard coordinates on $\mathbb{R}^3$ then define a membership function on M by $\mu(x, y, z) = y^2 + z^2$. Thus $\mu = 1$ on the equator $x = 0$, and $\mu = 0$ at the poles $(1, 0, 0)$ and $(-1, 0, 0)$. Note this cannot be represented with the usual notions of fuzzy numbers. Let f_2 be the function on M given by $f_2(x, y, z) = x^2 + y^2$. The induced membership in this case is easy to compute. The range of f_2 is the interval $\mathbf{I} = [0, 1]$. For any $\xi \in \mathbf{I}$, $\mu_{f_2}(\xi) = 1$ since $f_2^{-1}(\xi)$ will contain at least one point (x, y, z) on M where $x = 0$. At this point $\mu(x, y, z) = 1$, and so when the supremum is computed, 1 will be the result. Thus $\mu_{f_2}(\xi) = 1$ when $\xi \in \mathbf{I}$ and $\mu_{f_2}(\xi) = 0$ otherwise. This example will also receive more attention in Section 2.

2 Membership Induced by Lie Differentiation

In this section a definition is given for the membership induced by the derivative of a function along the integral curves of a vector field. The set of real-valued (C^∞, by hypothesis)

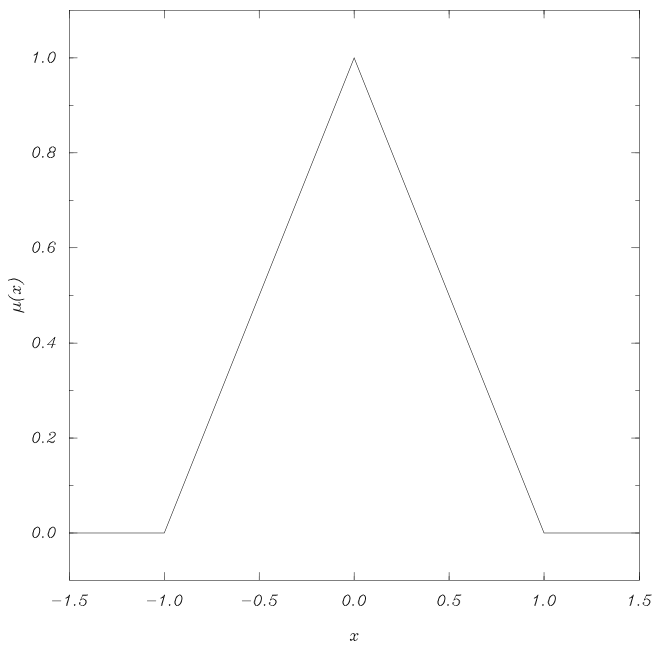

Figure 1: Membership on IR for x.

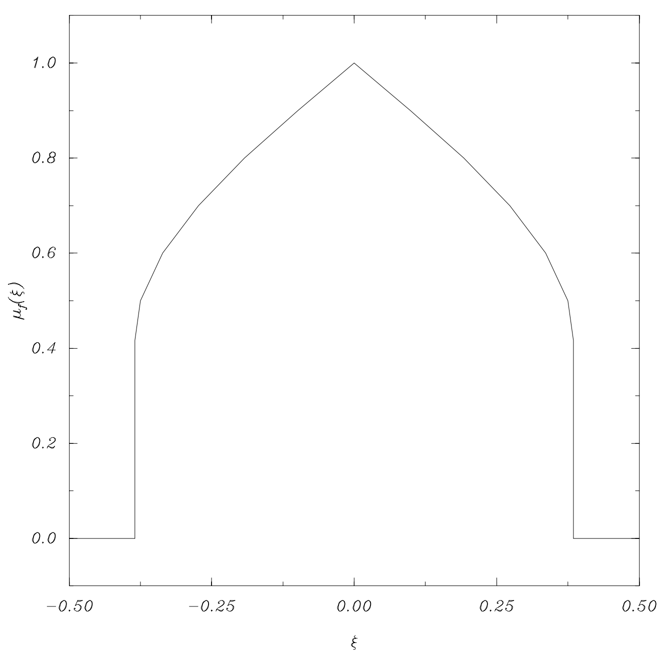

Figure 2: Membership induced on IR by f_1.

functions on a manifold M will be denoted by $\mathcal{F}(M)$, and the set of vector fields on M will be denoted by $\mathcal{X}(M)$.

Let $X \in \mathcal{X}(M)$ and $f \in \mathcal{F}(M)$. The *Lie derivative* (see Lang [7]) of f along X will be written $\mathcal{L}_X f$. If $(x^1, \ldots, x^n)$ are coordinates in some chart for M, and if, in this chart, the vector field X is represented by $(X^1(x^1, \ldots, x^n), \ldots, X^n(x^1, \ldots, x^n))$ (*i.e.*, $X = X^i \partial/\partial x^i$, with summation over repeated indices), then

$$\mathcal{L}_X f = X^i \frac{\partial f}{\partial x^i} \tag{2.1}$$

with summation over repeated indices. Thus, for every $X \in \mathcal{X}(M)$, the Lie derivative assigns a function $\mathcal{L}_X f$ to every $f \in \mathcal{F}(M)$. In fact, as can easily be seen from (2.1), $\mathcal{L}_X$ is a derivation on the $\mathbb{R}$-algebra $\mathcal{F}(M)$. Thus memberships can be computed for $\mathcal{L}_X f$ in the manner given in Definition 1.1. This leads to the following definition.

2.1 Definition: The *membership induced by f along X* is the membership induced on $\mathbb{R}$ by $\mathcal{L}_X f$. This will be denoted by

$$\mu_{X,f}(\eta) = \sup\{\mu(x) \mid x \in M, \quad \mathcal{L}_X f(x) = \eta\}$$

where $\eta \in \mathbb{R}$. If $(\mathcal{L}_X f)^{-1}(\eta) = \emptyset$, take $\mu_{X,f}(\eta) = 0$.

As a special case of this definition, let $M = \mathbb{R}^n$, and let $(x^1, \ldots, x^n)$ be the standard coordinates for M. Then there are n distinguished vector fields on M given by $X_i = \partial/\partial x^i$, for $i = 1, \ldots, n$. For a function $f : \mathbb{R}^n \to \mathbb{R}$, $\mu_{X_i,f}$ will give the membership on $\mathbb{R}$ induced by the function $\partial f/\partial x^i$ on $\mathbb{R}^n$. In particular, if $n = 1$, then Definition 2.1 reduces to the definition of differentiation on $\mathbb{R}$ as presented in [3] by Dubois and Prade.

Thus membership induced through differentiation is now made clear for more general cases than simple real numbers. Recall that the purpose in doing so was to observe the membership on the performance variables represented by the functions and their derivatives. If the resulting memberships are observed independently, misleading results might be inferred. This is because in order to specify the derivative, one must also specify the point at which it is evaluated. When observing the membership for values of the derivative, there is an associated value of the function itself. Thus one must consider them simultaneously.

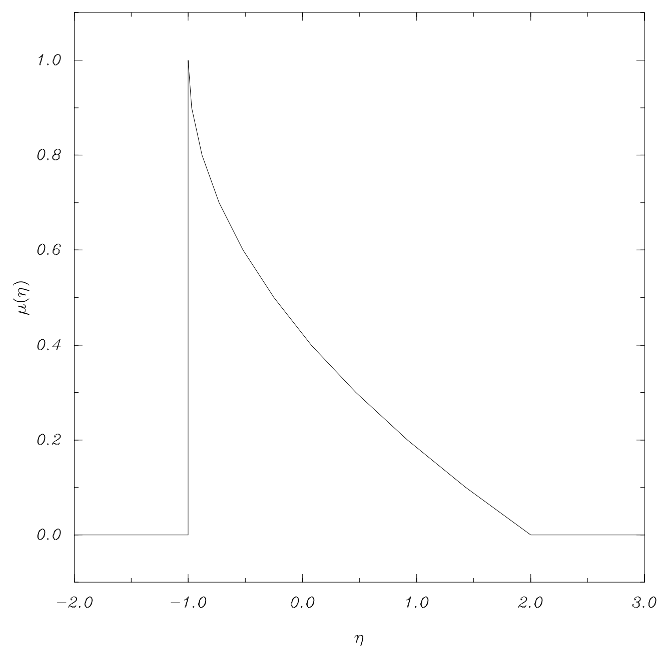

Figure 3: Membership induced on $\mathbb{R}$ by $\mathcal{L}_{X_1} f_1$.

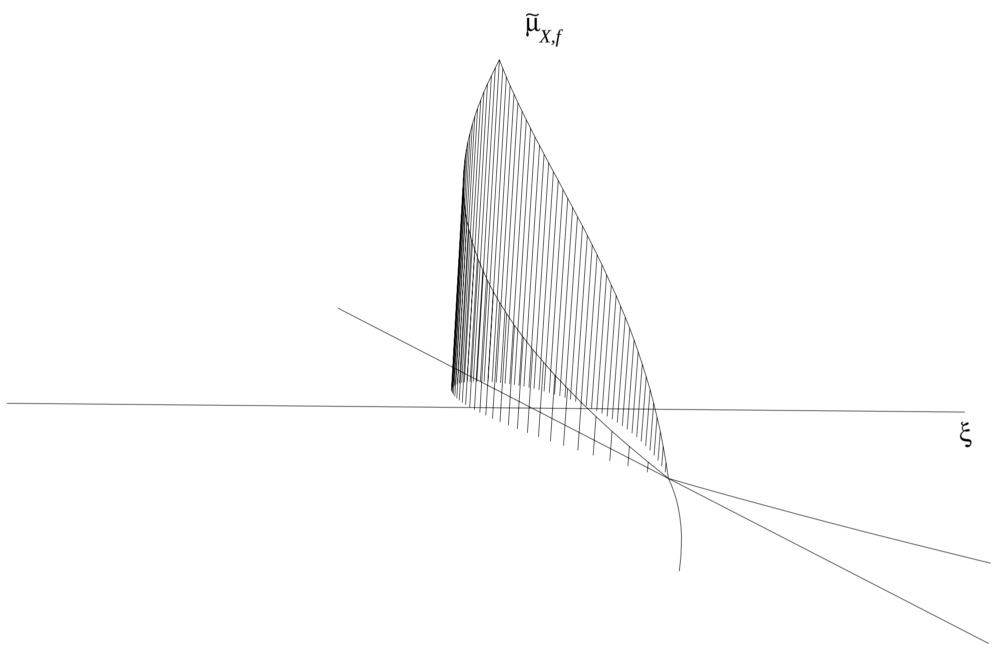

Figure 4: Membership induced on $\mathbb{R}^2$ by f_1 and $\mathcal{L}_{X_1} f_1$.

We now develop this formally. Given a function f and a vector field X on a manifold M, a membership on $\mathbb{R}^2$ can be defined which represents the membership induced by f along with the membership induced by f along X. More precisely, a membership is induced on $\mathbb{R}^2$ as follows.

$$\begin{aligned} \tilde{\mu}_{X,f} \quad &: \quad \mathbb{R}^2 \to [0,1] \\ &(\xi, \eta) \mapsto \sup\{\mu(x) \mid x \in M, \quad f(x) = \xi, \quad \mathcal{L}_X f(x) = \eta\} \end{aligned} \tag{2.2}$$

As usual, if $f^{-1}(\xi) \cap \mathcal{L}_X f^{-1}(\eta) = \emptyset$, then $\tilde{\mu}_{X,f}(\xi, \eta) = 0$. This is an example of Definition 1.1 where $N = \mathbb{R}^2$.

To illustrate the use of (2.2), the examples of Section 1 are treated again with the addition of differentiation along specified vector fields.

2.2 Example: Let $M = \mathbb{R}$ and define f_1 as in Example 1.2. As a membership function on M, take the function whose graph is depicted in Figure 1. Let $X_1 = \partial/\partial x$ where x is the standard coordinate for $\mathbb{R}$. A quick calculation gives $\mathcal{L}_{X_1} f_1 = 3x^2 - 1$. The membership induced on $\mathbb{R}$ by $\mathcal{L}_{X_1} f_1$ is shown in Figure 3. However, the memberships on $\mathbb{R}$ by both f_1 and $\mathcal{L}_{X_1} f_1$ cannot each be considered independently, for example, when selecting values for f_1 and $\mathcal{L}_{X_1} f_1$. Figure 4 shows the membership on $\mathbb{R}^2$ induced by f_1 and $\mathcal{L}_{X_1} f_1$ as given by (2.2). Observe that if the curve representing non-zero membership is projected onto the $(\xi, \tilde{\mu}_{X,f})$-plane, the membership induced by f as shown in Figure 2 results, and if the same curve is projected onto the $(\eta, \tilde{\mu}_{X,f})$-plane, the membership induced by f along X results, as shown in Figure 3. This observation is true in general, as can easily be seen by checking the definition.

2.3 Example: Let M, μ, and f_2 be as in Example 1.3. Define a vector field on $\mathbb{R}^3$ by $X_2 = y\partial/\partial x - x\partial/\partial y$. Thus X_2 generates a uniform rotation about the z-axis in $\mathbb{R}^3$. It is easily verified that X_2 leaves M invariant, and so defines a vector field on M by restriction, which we will also denote by X_2. The integral curves of X_2 on M are shown in Figure 5. In Figure 6 the membership on $\mathbb{R}^2$ is plotted as computed by (2.2) given the membership μ on M. Observe that the projection of the membership curve onto the $(\eta, \tilde{\mu}_{X,f})$-plane consists of a single point. This reflects the fact that f_2 is constant along X_2 (*i.e.*, $\mathcal{L}_{X_2} f_2 = 0$).

2.4 Example: Let M, μ, and X_2 be the same as in Example 2.3. Define a function f_3 on M by $f_3(x, y, z) = y^2 + z^2$. The membership on $\mathbb{R}^2$ as computed by (2.2) is shown in

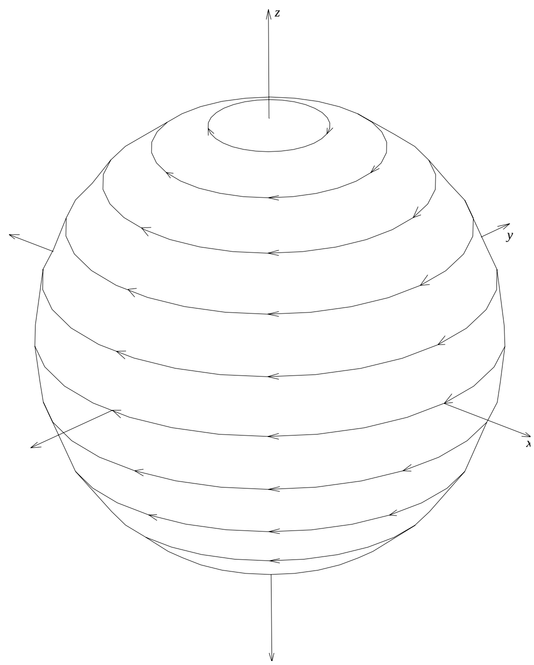

Figure 5: Integral curves for X_2 on S^2.

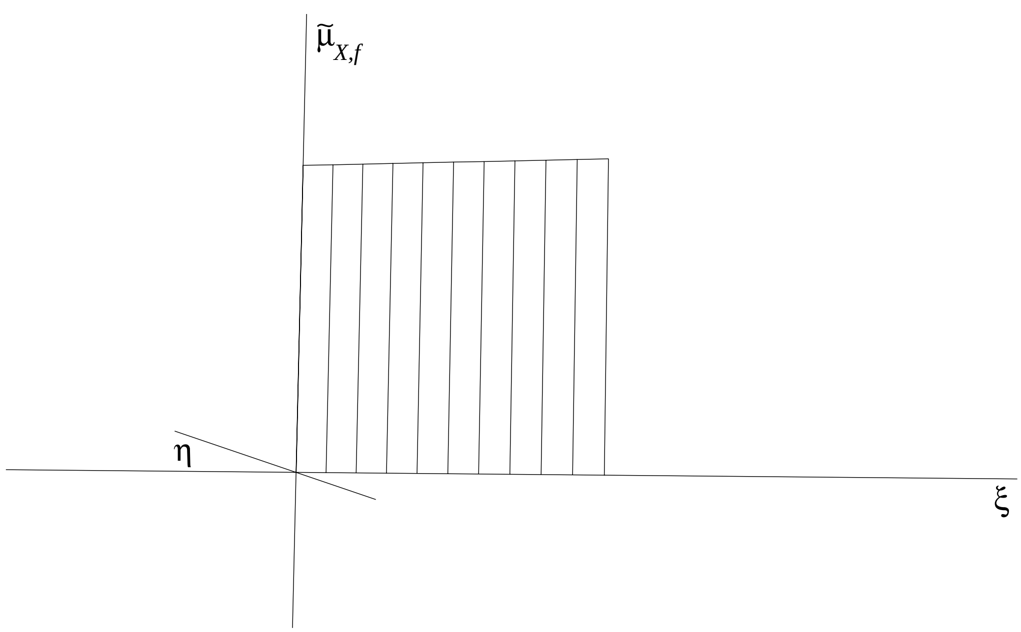

Figure 6: Membership induced on $\mathbb{R}^2$ by f_2 and $\mathcal{L}_{X_2} f_2$.

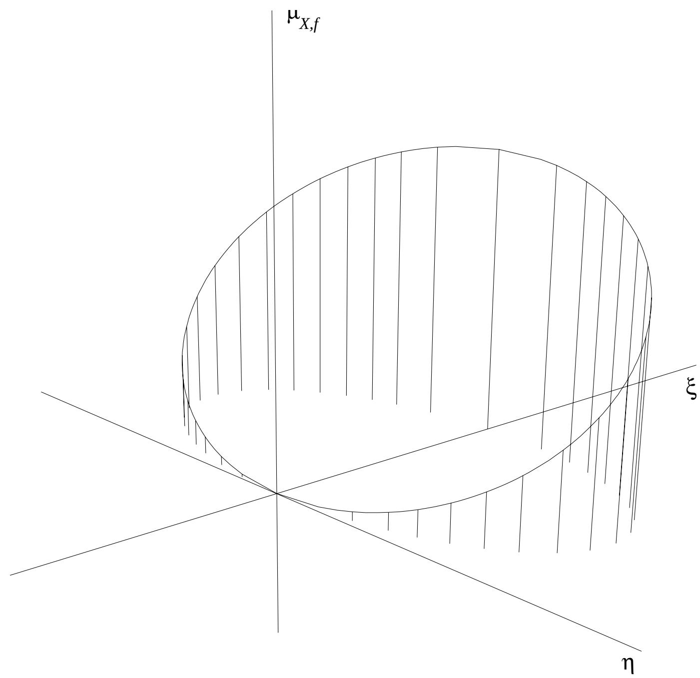

Figure 7: Membership induced on $\mathbb{R}^2$ by f_3 and $\mathcal{L}_{X_2} f_3$.

Figure 7. A straightforward calculation gives $\mathcal{L}_{X_2} f_3 = -2xy$. Observe that f_3, unlike f_2, is not constant along integral curves of X_2. This is verified by the fact that the projection of the membership curve in Figure 7 onto the $(\eta, \tilde{\mu}_{X,f})$-plane is nontrivial. Also note that the projection of the membership curve onto the $(\xi, \tilde{\mu}_{X,f})$-plane yields a triangular membership function increasing linearly from 0 to 1. This reflects the fact that f_3 has been chosen to be the same as the membership μ.

3 Membership Induced by Flows of Vector Fields

In this section, ordinary differential equations (o.d.e.'s) will be presented with one, or both, of the following effects of imprecision.

1. How membership on the variables gets modified as the variables evolve according to the flow of the o.d.e., and
2. the existence of imprecision in parameters in the o.d.e.

A means of computing memberships induced by the flow in these instances is presented. Cases $1)$ and $2)$ are examples of a more general problem which is formulated first. In keeping with the spirit of the first two sections, o.d.e.'s are formulated in the language of vector fields on a manifold.

As in the previous sections, M will be a smooth manifold. Since vector fields may depend on parameters, a manifold P is needed which will be the *parameter space*. Let $\pi_M : M \times P \to M$ be projection onto the first factor. The *tangent bundle* of M will be denoted by TM and $\tau_M : TM \to M$ will denote the tangent bundle projection. As in Lang [7], the *pull-back bundle* $\pi_{MP} : \pi_M^* TM \to M \times P$ can be formed, where

$$\pi_M^* TM = \{(v, (x, p)) \in TM \times (M \times P) \mid \tau_M(v) = \pi_M(x, p)\}$$

and

$$\pi_{MP}(v, (x, p)) = (x, p)$$

The manifold $\pi_M^* TM$ is to be regarded as a vector bundle over $M \times P$. The fibre over a point $(x, p) \in M \times P$ is $\pi_{MP}^{-1}(x, p) = T_x M$. Now we can make a definition.

3.1 Definition: A *vector field with parameters* on M is a smooth section $X : M \times P \to \pi_M^* TM$ of $\pi_M^* TM$ such that for every $p \in P$, $X_p : M \to TM : x \mapsto X(x, p)$ is a vector field on M.

If X_p is the vector field on M corresponding to the parameter $p \in P$, its *flow* will be the one-parameter family of diffeomorphisms of M which will be denoted by $F_t^p : M \to M$. All vector fields considered here will be complete.

Now suppose a membership, μ, is given on $M \times P$. For each $t \in \mathbb{R}$, a membership on M induced by a vector field with parameters can be defined as follows.

$$\begin{aligned} \mu_t \quad : \quad & M \to [0, 1] \\ & x \mapsto \sup\{\mu(x', p) \mid (x', p) \in M \times P, \quad F_t^p(x') = x\} \end{aligned} \tag{3.1}$$

If $\bigcup_{p \in P} F_{-t}^p(x) = \emptyset$, then take $\mu_t(x) = 0$. Negoiţă and Ralescu [8] discuss a similar notion in the context of iterating a set map $f : X \to X$ which depends on imprecise parameters. Novàk [9] extends these ideas to include cases where imprecise initial conditions are mapped under the set mapping f. This situation is included in the definition (3.1).

The cases $1)$ and $2)$ can be thought of as special cases of (3.1) as follows.

1. *Crisp parameters:* In this case, an initial membership function, μ_M, is given on M. Since there is no dependence on parameters, P is taken to be the manifold consisting of a single point p so that we have the single vector field X_p on M with flow F_t^p. Now define a membership μ on $M \times P$ by $\mu(x,p) = \mu_M(x)$. Applying (3.1) will determine how the membership μ_M will evolve under the flow of X_p. See 3.1 below.

2. *Imprecise parameters:* In this case, a point $x_0 \in M$ is fixed and a membership on P gives rise to an induced membership on M as x_0 gets mapped under the flow of the vector field for the various parameter values. Suppose μ_P is some given membership on P. A membership can be defined on $M \times P$ by $\mu(x,p) = \mu_P(p)$ if $x = x_0$ and $\mu(x,p) = 0$ otherwise. Computing the induced membership given by (3.1) will give the desired membership on M for each t. See 3.2 below.

3.1 Example: In this example, the first case is illustrated. Let $M = \mathbb{R}^2$ and let (x,y) be the standard coordinates for M. Define a vector field $X_4 = -x\partial/\partial x + y\partial/\partial y$ on M. Such a vector field is representative of a two dimensional ideal fluid flow against a wall perpendicular to the flow. The flow of this vector field is given by the one-parameter family of diffeomorphisms $F_t^4 : M \to M : (x,y) \mapsto (xe^{-t}, ye^t)$. Figure 8 shows how two particular membership functions on M starting at t_0 and t'_0 are mapped by the flow to memberships on M at times t_1, t_2, and t'_1, t'_2, respectively. This is a very simple example. More complicated flows will give rise to more distorted induced memberships.

3.2 Example: In this example, the second case is illustrated. Let $M = \mathbb{R}$ with x the standard coordinate on M. As a vector field on M take $X_5 = ax\partial/\partial x$. Such a vector field represents the growth of an exponential process. Here $a \in \mathbb{R}$ is a parameter and has the same membership function as was used for x in Example 1.2. See Figure 1. The flow of X_5 is given by $F_t^5 : \mathbb{R} \to \mathbb{R} : x \mapsto xe^{at}$. In Figure 9 the membership μ_t on M is computed for two times, t_1 and t_2. The initial point $x_0 \in \mathbb{R}$ has been chosen to be positive. Observe that at $t = 0$, the induced membership on M has support $\{x_0\}$.

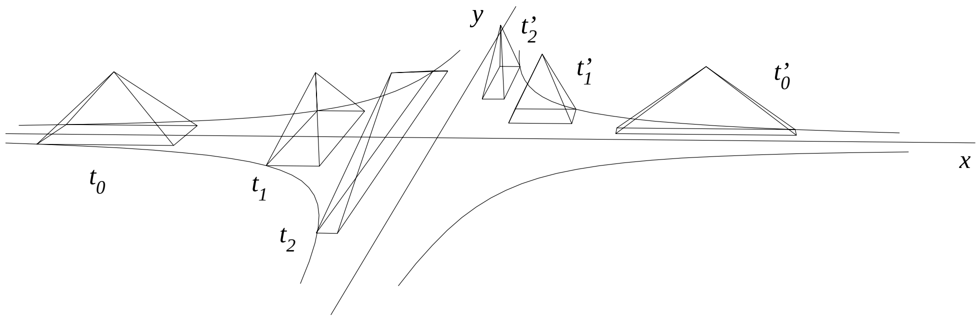

Figure 8: Membership induced by the flow of X_4.

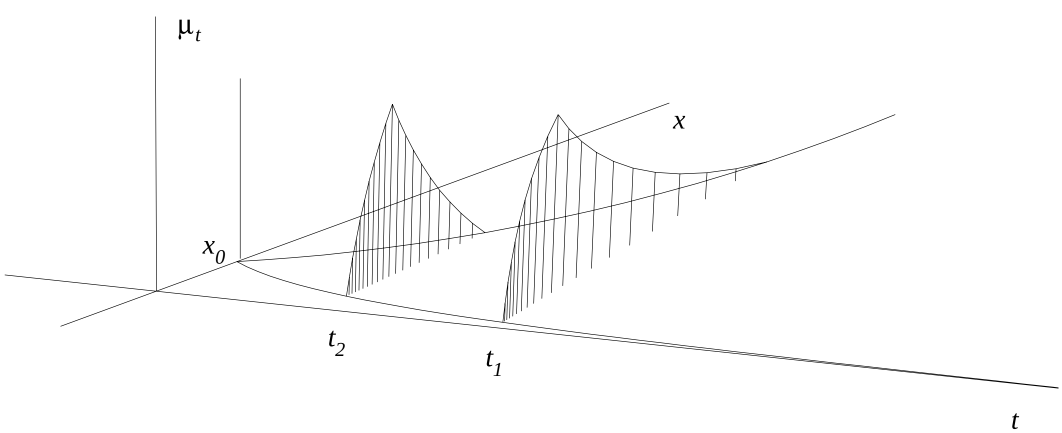

Figure 9: Membership induced on $\mathbb{R}$ by the flow of X_5 with membership on a.

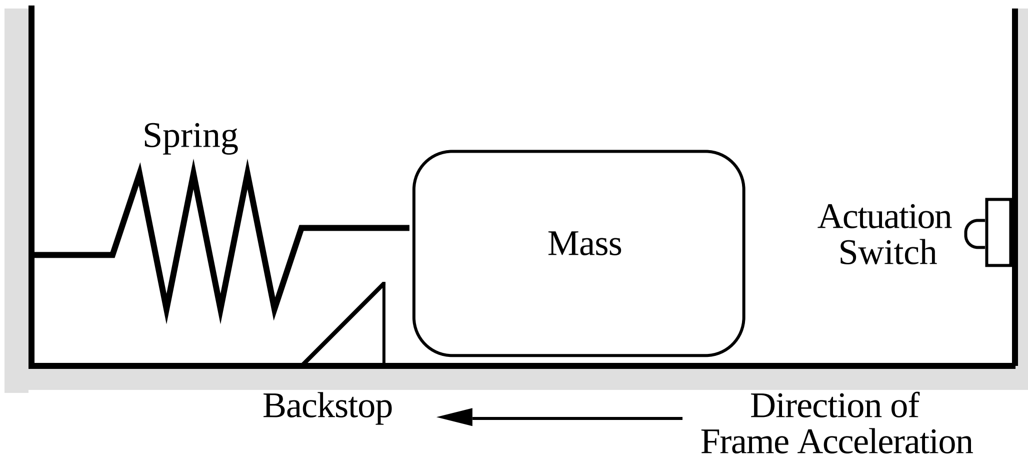

Figure 10: Accelerometer design.

4 Engineering Example: Design of an Accelerometer Switch

As an industrial example of using induced membership through imprecise o.d.e's, consider the design of a uni-directional accelerometer, which indicates accelerations above a threshold with a switch closure. It can be modeled as a simple mass spring system, as shown in Figure 10. Under specified accelerations, the accelerometer mass must contact a switch within specified time durations. Imprecision in this model reflects that, at the start of a design process, the designer does not know what parameter values are desired to be used. Typically, though, the designer may be able to state that certain values of a design parameter will not work, and that some may work better than others. Furthermore, the designer may adjust values of the design parameters to better satisfy the customer, based on the performance evaluations. Thus there is informal interpretation among the parameter values.

In the accelerometer design (shown in Figure 10) there is a mass m attached to a spring k attached to the ground. The ground is accelerated. With sufficient acceleration, the mass must displace a specified distance to make contact with a switch. There is also a backstop placed against the mass. Refer to Figure 11.

To determine the time to actuate the switch, the differential equation of motion of the mass must be solved. It is:

$$m(\frac{d^2x}{dt^2} + a) + kx = 0 \tag{4.2}$$

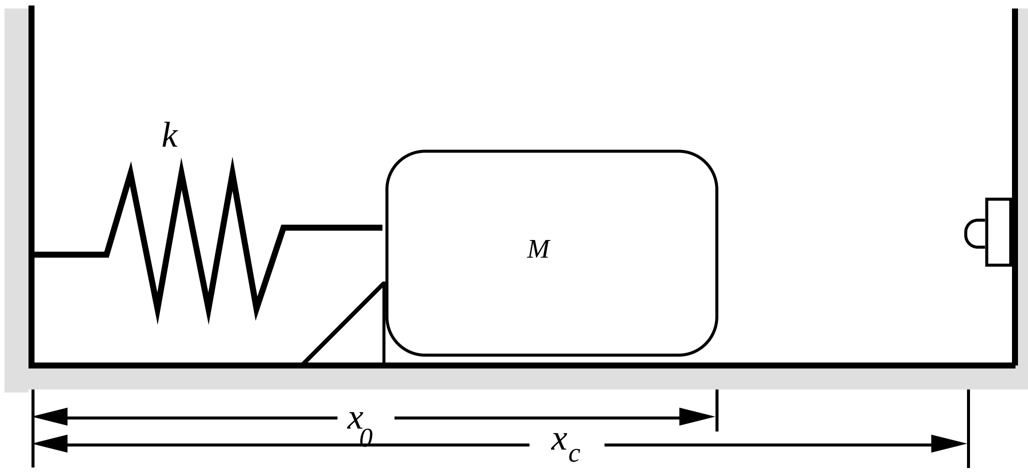

Figure 11: Accelerometer model.

with $x(0) = x_0$ and $\dot{x}(0) = 0$.

Imprecision exists in the model, in that the designer does not initially know what values of mass m, spring constant k, and backstop position x_0 should be used. Subjectively, however, the designer may feel appropriate values are as specified by the membership functions shown in Figure 12.

The manifold in this case is $M = \mathbb{R}$, and the additional parameter space is $P = \mathbb{R}^2$, with $p = (m, k) \in P$. A membership function $\mu(x_0, m, k)$ on $M \times P = \mathbb{R}^3$ can be defined as

$$\mu(x_0, m, k) = \min\{\mu(m), \mu(k), \mu(x_0)\} \tag{4.3}$$

With this definition, (4.2) becomes an imprecise o.d.e. with imprecise starting conditions and imprecise parameters. Applying (3.1), the induced preference μ_t, where

$$\mu_t(x) = \sup\{\mu(x_0, m, k) \mid (x_0, m, k) \in M \times P, F_t^p(x_0) = x\}, \tag{4.4}$$

can be graphed on the phase space $(x, \dot{x})$ for different times t, as shown in Figure 13. The initial imprecision of x_0, as shown at $t = 0$, is spread out across the phase space as time elapses, due to the imprecision in the o.d.e. from the imprecision of m and k. Thus, the designer has much freedom in attaining values of $(x, \dot{x})$ at any time to actuation t.

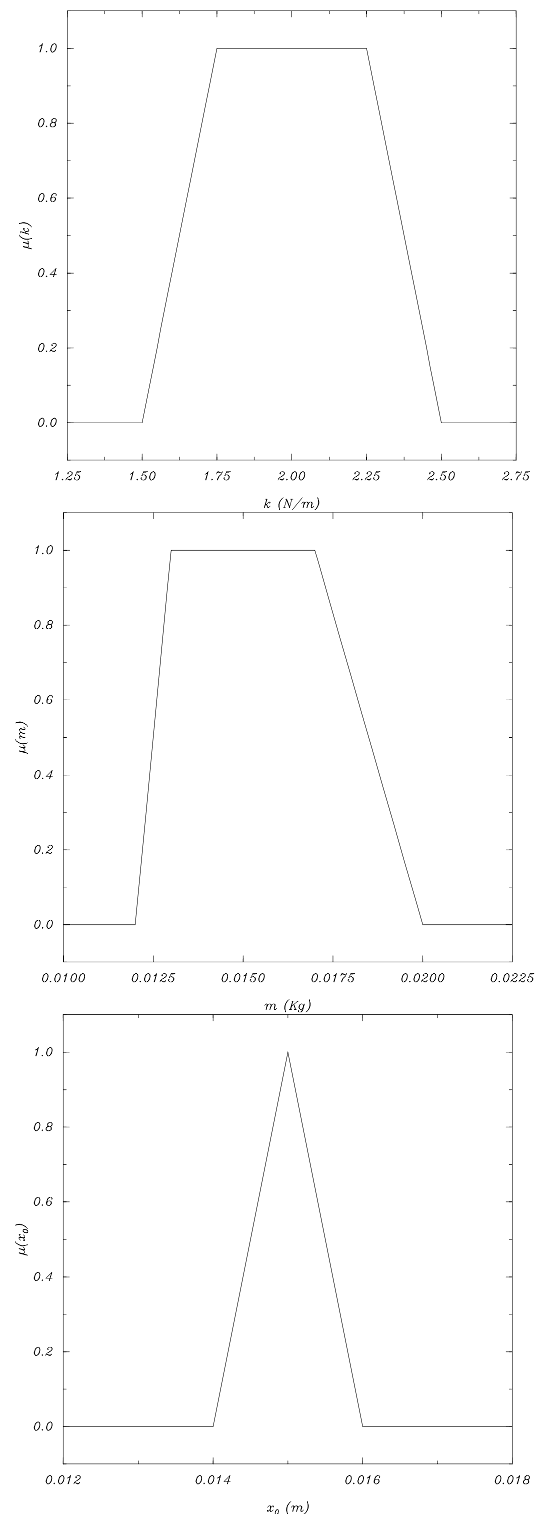

Figure 12: m, k, and x_0 preferences.

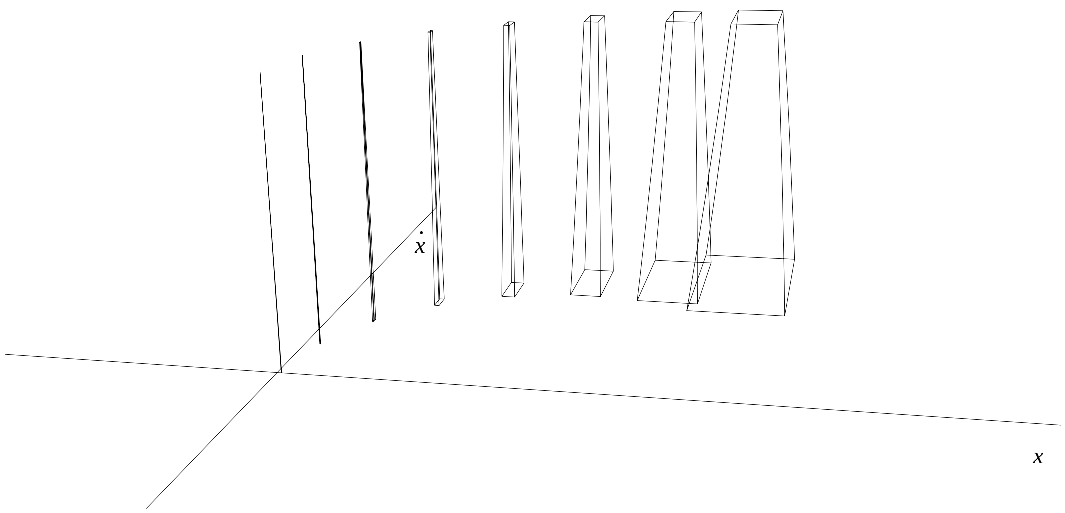

Figure 13: Phase space of the accelerometer.

5 Conclusions and Future Work

Methods have been presented for propagating membership on manifolds through the operation of Lie differentiation, and by the flow of vector fields which may or may not depend on imprecise parameters.

The formula for induced membership on $\mathbb{R}^2$ given by (2.2) seems to merit further exploration. In Example 2.3 the membership for the derivative of a function which is constant along a vector field was demonstrated to have support equal to $\{0\}$. It would be interesting to see if the topology of the manifold has any effect on the possible distribution of membership on $\mathbb{R}^2$. For example, is it true that a compact manifold will give rise to a compactly supported membership? Figure 7 seems to suggest that this may indeed be the case.

Interpretations for memberships induced by the flows of vector fields seem to be more readily made. Figures 8 and 9 agree with what is expected intuitively. Namely, in Figure 8 the initial membership on $\mathbb{R}^2$ is simply transported along the integral curves of the vector field, and in Figure 9, an initial membership concentrated at the point x_0 is spread out over $\mathbb{R}$ as time elapses. However, even for examples of this type, some rather complicated behaviour can emerge. For example, if the chosen range of parameters includes a *bifurcation value*, the qualitative behaviour of the induced membership could change drastically depending on the parameter.

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