Approximating α-cuts With the Vertex Method

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Abstract

If $f : \mathrm{R}^n \rightarrow \mathrm{R}$ is continuous and monotonic in each variable, and if μ_i is a fuzzy number on the i^{th} coordinate, then the membership on R induced by f and by the membership on R^n given by $\mu(x) = \min(\mu_1(x^1), \ldots, \mu_n(x^n))$ can be evaluated by determining the membership at the endpoints of the level cuts of each μ_i. Here more general conditions are given for both the function f and the manner in which the fuzzy numbers $\{\mu_i\}$ are combined so that this simple method for computing induced membership may be used. In particular,

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a geometric condition is given so that the α-cuts computed when the fuzzy numbers are combined using *min* is an upper bound for the actual induced membership.

Keywords: Fuzzy numbers; analysis; topology; data analysis methods; multiple criteria evaluation; engineering

Introduction

In this paper, we consider the conditions required on different combinations of membership functions to result in the same induced membership on a dependent variable. As motivation for this work, consider a simple example. Two variables $(x, y) \in \mathrm{R}^2$ are mapped to R by $f(x, y) = xy$. Suppose there are fuzzy numbers X and Y, where $\mu(x)$ is a triangular membership function with $\mathrm{supp}(\mu) = [0, 1]$ and a peak at $x = \frac{1}{2}$. Here $\mathrm{supp}(\mu)$ denotes the support of μ. Likewise, suppose $\mu(y)$ is a triangular membership function with $\mathrm{supp}(\mu) = [0, 2]$ and a peak at $y = 1$. Using the extension principle definition, one can define an induced membership onto $z \in \mathrm{R}$ by

$$\mu_f(z) = \sup\{\min\{\mu(x), \mu(y)\} \mid z = xy\}.$$

Since $f(x, y) = xy$ is continuous monotonic in x and y, it is well known that $\mu_f(z)$ can be evaluated by determining the membership at the endpoints of the level cuts of $\mu(x)$ and $\mu(y)$ [5, 7, 8, 9, 10, 11]. Doing so results in an induced membership $\mu_f(z)$ as shown in Figure 1.

Now consider a problem where compensation is desired in the induced membership definition. Perhaps we wish to know the induced membership through a definition of

$$\mu'_f(z) = \sup\{(\mu(x)\mu(y))^{\frac{1}{2}} \mid z = xy\}.$$

Such might be the case when X and Y are parameters in an imprecisely known

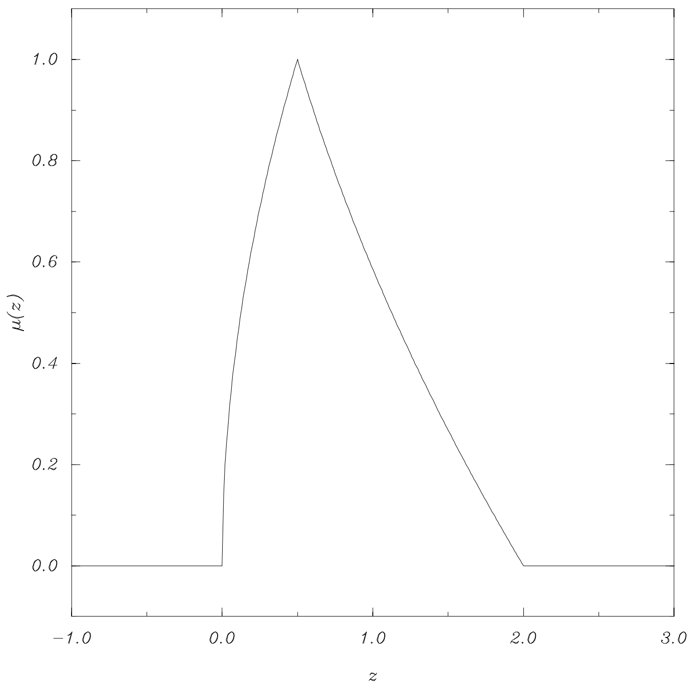

Figure 1: Induced membership through $z = xy$.

engineering model whose performance we desire to be simulated [12, 13, 14]. When the induced preference $\mu'_f(z)$ is calculated, we find that $\mu'_f(z) = \mu_f(z)$ for all z. Thus $\mu'_f(z)$ could be determined by the simpler vertex method, and the result is invariant. General conditions under which this is true are explored in this paper.

By a *membership function*, we mean a map μ from a set X to the interval $[0, 1]$. By an α-cut of μ we mean the set of all points $x \in X$ such that $\mu(x) \geq \alpha$. A certain type of membership function can be associated to a *fuzzy number* [6]. In the sequel, whenever we consider a fuzzy number, its membership function is assumed to have compact support. Therefore, the membership function for a fuzzy number will be a map, μ, from R to the interval $[0, 1]$ such that:

1. $\text{supp}(\mu) = [x_{min}, x_{max}]$, and

2. there exists $x_* \in (x_{min}, x_{max})$ such that $\mu(x_*) = 1$, and μ is continuous and monotonically increasing on $[x_{min}, x_*]$ and is continuous and monotonically

decreasing on $[x_*, x_{max}]$.

This is equivalent to the usual definition [6] of a fuzzy number as a convex and normal fuzzy set over R.

Suppose that we have a fuzzy number on each of the n coordinates in R^n. Then it is possible to assign a membership, μ, on R^n by defining

$$\mu(x^1, \ldots, x^n) = \min(\mu_1(x^1), \ldots, \mu_n(x^n)) \tag{0.1}$$

Then, given a function $f : \mathrm{R}^n \to \mathrm{R}$, it is possible to define the membership induced by f on R by the extension principle. Thus

$$\mu_f(y) = \sup\{\mu(x) \mid f(x) = y\} \tag{0.2}$$

As usual, the convention is adopted that $\mu_f(y) = 0$ if $f^{-1}(y) = \emptyset$. Given this definition of induced membership, it is desirable to find as simple a means as possible for computing the membership at a point $y \in \mathrm{R}$. It is well-known [5, 7, 8, 9, 10, 11] that if f is continuous and monotonic, then one need only know the value of the function at the endpoints of the α-cuts for each μ_i in order to determine the endpoints for the α-cut interval for μ_f. This is the vertex method discussed in Section 2.

In this paper we extend these computational ideas to more general functions f and more general ways of combining the membership functions $\{\mu_i\}$ to get a membership function on R^n. In particular we determine geometric conditions on the level sets of f and μ sufficient to determine if the resulting α-cut is contained in the α-cut computed using *min* as a combination function.

In the first section, we explore a generalization of the ideas given in Buckley and Qu [1] for computing the induced membership when f is arbitrary, and the memberships $\{\mu_i\}$ are combined using *min*. In particular, we observe that the results in [1] follow from the nested nature of the α-cuts. So the arguments in [1] are immediately extendable to a more general class of problems. In Section 2, a simple algorithm, the vertex method, is discussed for computing the α-cuts on R for the membership induced by a map $f : \mathrm{R}^n \to \mathrm{R}$. The applicability of this algorithm to various types of functions and combination functions is explored in Section 3. In particular, geometric hypotheses are given on the level sets of the function and the combination function so that the α-cuts determined by the vertex method contain those for the actual induced membership.

1 Computing Induced Membership Using α-cuts

In this section we present generalizations of Theorems 1 and 2 in [1]. For the moment, we will only consider the case where μ is a membership function on R^n determined by Equation (0.1). We will also consider f to be an *arbitrary* function. As notation let

$$\mathcal{X}^i_\alpha = \{x^i \in \mathrm{R} \mid \mu_i(x^i) \geq \alpha\} \tag{1.1}$$

Since $\{\mu_i\}$ are fuzzy numbers, $\mathcal{X}^i_\alpha$ is a closed interval, say $\mathcal{X}^i_\alpha = [x^i_{min}(\alpha), x^i_{max}(\alpha)]$. If $\alpha = 0$, we take the convention that $\mathcal{X}^i_0 = \mathrm{supp}(\mu_i)$.

Buckley and Qu [1] give two equivalent ways to compute the induced membership. One is as given by Equation (0.2) and the other is defined as follows

$$\nu_f(y) = \sup\{\alpha \mid y \in f(\mathcal{X}^1_\alpha \times \cdots \times \mathcal{X}^n_\alpha)\} \tag{1.2}$$

One thus wishes to show that

$$\mu_f(y) = \nu_f(y) \quad \text{for all} \quad y \in \mathrm{R}. \tag{1.3}$$

Upon close examination of this statement, it becomes clear that the key ingredient in its proof is the fact that $\mu^{-1}(\alpha)$ ($\mu^{-1}(\beta)$ if $\alpha > \beta$. In this case, the essential elements of what is needed for the proof of Equation (1.3) are contained in the following proposition.

1.1 Proposition: *Let X be a set, and let $\{X_\alpha \subset X \mid \alpha \in [0,1]\}$ be a family of subsets which satisfy the inclusion relation X_α (X_β for $\alpha > \beta$. Let $f : X \to Y$ be* *a set map. For $y \in Y$ define*

$$\Theta(y) = \sup\{\alpha \mid x \in X_\alpha,\ f(x) = y\}$$

and

$$\Phi(y) = \sup\{\alpha \mid y \in f(X_\alpha)\}$$

Then $\Theta(y) = \Phi(y)$ for all $y \in Y$.

Proof: Our proof follows that in [1] closely. Let $y \in Y$ be such that $\Phi(y) = \alpha_* > 0$ and let $\epsilon > 0$. Then there exists $\beta \in (\alpha_* - \epsilon, \alpha_*)$. Since

$$\Phi(y) = \sup\{\alpha \mid y \in f(X_\alpha)\},$$

$y \in f(X_\beta)$. Therefore, there exists $x \in X_\beta$ such that $f(x) = y$. Since $X_\beta \subsetneq X_\alpha$ when $\beta > \alpha$ we may conclude that

$$\Theta(y) = \sup\{\alpha \mid x \in X_\alpha,\ f(x) = y\} \geq \beta$$

Thus $\Theta(y) \geq \beta \geq \alpha_* - \epsilon$. But, since $\epsilon > 0$ is arbitrary, $\Theta(y) \geq \alpha_* = \Phi(y)$.

Now let $\Theta(y) = \alpha_* > 0$, and let $\epsilon > 0$. As before, there exists $\beta \in (\alpha_* - \epsilon, \alpha_*)$. Since

$$\Theta(y) = \sup\{\alpha \mid x \in X_\alpha,\ f(x) = y\}$$

and since the inclusion relation holds, we may conclude that $y \in f(X_\beta)$. Thus

$$\Phi(y) = \sup\{\alpha \mid y \in f(X_\alpha)\} \geq \beta$$

so we have $\Phi(y) \geq \beta \geq \alpha_* - \epsilon$. Again, since $\epsilon > 0$ is arbitrary, we get $\Phi(y) \geq \alpha_* = \Theta(y)$.

The last two paragraphs prove that $\Theta(y) = \Phi(y)$ if both quantities are positive. Now if we have $\Phi(y) = 0$ or $\Phi(y) = 0$ then the other is clearly zero since the inclusion relation holds. ■

As a corollary to Proposition 1.1 we can consider the case when the nested sets, X_α, are the level sets of a membership function.

1.2 Corollary: *Let X and Y be sets and let $f : X \to Y$ be a map. Let $\mu : X \to [0,1]$ be an membership function on X, and let X_α denote the α-cut of μ. Suppose that the inclusion relation $X_\alpha \subsetneq X_\beta$ for $\alpha > \beta$ holds. For $y \in Y$ define*

$$\mu_f(y) = \sup\{\mu(x) \mid x \in X, f(x) = y\}$$

and

$$\nu_f(y) = \sup\{\alpha \mid y \in f(X_\alpha)\}$$

Then $\mu_f(y) = \nu_f(y)$ for all $y \in Y$.

Proof: First note the following

$$\begin{aligned}\mu_f(y) &= \sup\{\mu(x) \mid x \in X, f(x) = y\} \\ &= \sup\{\alpha \mid x \in X_\alpha, f(x) = y\}\end{aligned}$$

Now, since the inclusion relation holds, the hypotheses of Proposition 1.1 are satisfied. ■

This provides a generalization of Theorem 1 in Buckley and Qu [1] where the special cases $X = \mathrm{R}^n$ and $Y = \mathrm{R}$ are considered. It is also possible to generalize Theorem 2 in [1] to topological spaces as follows.

1.3 Proposition: *In Proposition 1.1, suppose X and Y are topological spaces, and f is a continuous map. Suppose that $\{X_\alpha\}$ are as in Proposition 1.1 and that X_0 is compact. Let W_α and Y_α be the α-cuts of Θ and Φ, respectively. Then $W_\alpha = Y_\alpha = f(X_\alpha)$.*

Proof: From Proposition 1.1 we know that $W_\alpha = Y_\alpha$. We will show that $Y_\alpha = f(X_\alpha)$.

Let $y \in f(X_{\alpha_*})$ so that $\Phi(y) = \sup\{\alpha \mid y \in f(X_\alpha)\} \geq \alpha_*$. Thus $y \in Y_{\alpha_*}$.

Let $y \in Y_{\alpha_*}$ so that $\Phi(y) = \beta_* \geq \alpha_*$. Let $\{\beta_n \in [0,1] \mid n \in \mathrm{N}\}$ be a sequence which converges to β_* and such that $\beta_n < \beta_*$ for all $n \in \mathrm{N}$. Since the inclusion relation holds, we have $y \in f(X_{\beta_n})$. Thus we can choose a sequence $\{x_n \in X \mid n \in \mathrm{N}\}$ such that $x_n \in X_{\beta_n}$ and $f(x_n) = y$ for all $n \in \mathrm{N}$. Since the sequence $\{x_n\}$ lies in X_0 which is compact, there exists a convergent subsequence $\{x_{n_k}\}$ converging to a value, say x_*. Since the sequence $\{\beta_n\}$ converges to β_*, we must have $x_* \in X_{\beta_*}$, and, since f is continuous, $f(x_*) = y$. Therefore $y \in f(X_{\alpha_*})$, since $\alpha_* \leq \beta_*$. ■

Observe that the inclusion $f(X_\alpha) \subseteq Y_\alpha = W_\alpha$ always holds.

2 The Vertex Method

In this section we discuss an algorithm, called the *vertex method* [2], to compute induced membership for certain problems [2, 3, 4, 7]. Throughout this section, μ_i is a fuzzy number on the i^{th} coordinate in R^n and f is a real-valued function on R^n. Recall from Section 1 the notation $\mathcal{X}^i_\alpha$.

2.1 Definition: Let $\alpha \in [0,1]$ and let $(x^1, \ldots, x^n) \in \mathcal{X}^1_\alpha \times \cdots \times \mathcal{X}^n_\alpha$. We say $(x^1, \ldots, x^n)$ is an *α-ridge point* if $(x^1, \ldots, x^n) = (x^1_{\beta_1}(\alpha), \ldots, x^n_{\beta_n}(\alpha))$ where $\beta_i \in \{min, max\}$ for $i = 1, \ldots, n$.

The set of all α-ridge points will be denoted by $\mathcal{R}_n(\alpha)$. Note that $|\mathcal{R}_n(\alpha)| = 2^n$. We wish to use this finite set to compute a closed interval $\mathcal{Y}_\alpha \subset \mathrm{R}$ which will be the proposed α-cut for the membership on R induced by f. We define the points:

$$y^\alpha_{min} = \min\{f(x) \mid x \in \mathcal{R}_n(\alpha)\}$$

$$y^\alpha_{max} = \max\{f(x) \mid x \in \mathcal{R}_n(\alpha)\}$$

in R and points:

$$x^\alpha_{min} = \{x \in \mathcal{R}_n(\alpha) \mid f(x) = y^\alpha_{min}\}$$

$$x^\alpha_{max} = \{x \in \mathcal{R}_n(\alpha) \mid f(x) = y^\alpha_{max}\}$$

in $\mathcal{X}^1_\alpha \times \cdots \times \mathcal{X}^n_\alpha$. If there is more than one such point x^α_{min} or x^α_{max}, we will choose one of them, without loss of generality.

2.2 Definition: The interval $[y^\alpha_{min}, y^\alpha_{max}]$ is called the *ridge point induced α-cut* and is denoted $\mathcal{Y}_\alpha$.

We will also have need of the following sets:

$$\mathcal{Y}^\alpha_{min} = \{y \in \mathrm{R} \mid y < y^\alpha_{min}\}$$

$$\mathcal{Y}^\alpha_{max} = \{y \in \mathrm{R} \mid y > y^\alpha_{max}\}$$

Thus $\mathcal{Y}^\alpha_{min} \cup \mathcal{Y}^\alpha_{max} = \mathrm{R} \setminus \mathcal{Y}_\alpha$.

3 Estimating α-cuts with the Vertex Method

In this section we will present conditions for the ridge point induced α-cuts to provide an upper or lower bound for the actual α-cuts. We also explore the effects of allowing combination functions more general than *min* for combining the membership functions $\{\mu_i\}$ to obtain a membership function on R^n. The most general means of combining the membership functions $\{\mu_i\}$ to be explored is the following:

3.1 Definition: Let μ_i be a fuzzy number on the i^{th} coordinate in R^n. A *combination function* is a continuous function $\mathcal{P} : \mathrm{I}^n \to \mathrm{I}$ so that the function $\mu(x^1, \ldots, x^n) = \mathcal{P}(\mu_1(x^1), \ldots, \mu_n(x^n))$ on R^n has its support contained in $\mathcal{X}^1_0 \times \cdots \times \mathcal{X}^n_0$. Here $\mathrm{I} = [0, 1]$.

Thus μ is a membership function on R^n which is zero at least whenever $\min(\mu_1(x^1), \ldots, \mu_n(x^n))$ is zero. In this section, whenever we write $\mu(x)$, we think of μ as being a membership function arising from a combination function. The α-cuts for the membership μ will be denoted by $\mathcal{X}_\alpha$. With this membership on R^n, we can define the induced membership μ_f on R for any function $f : \mathrm{R}^n \to \mathrm{R}$ as in Equation (0.2). The α-cuts of μ_f will be denoted by $\mathcal{W}_\alpha$. Thus $\mathcal{W}_\alpha \subset \mathrm{R}$. All combination functions conform to the following proposition.

3.2 Proposition: *Let $\mathcal{P}$ be a combination function with μ the corresponding membership function, and μ_f the membership induced on* R *by a continuous function $f : \mathrm{R}^n \to \mathrm{R}$. Let $\mathcal{X}_\alpha$ and $\mathcal{W}_\alpha$ denote the α-cuts for the membership functions μ and μ_f, respectively. Then $\mathcal{X}_\alpha$ and $\mathcal{W}_\alpha$ are compact for each $\alpha \in [0, 1]$.*

Proof: Since $\mathcal{P}$ is a combination function, the support of μ is contained in $\mathcal{X}_0^1 \times \cdots \times \mathcal{X}_0^n$, the support of *min*. Hence the support of μ is a closed subset of a compact set, and so is compact itself. Now fix $\alpha \in [0, 1]$. Then $\mathcal{X}_\alpha = \mu^{-1}([\alpha, 1])$. Observe that $[\alpha, 1]$ is a closed subset of $[0, 1]$. Also note that μ is continuous since $\mathcal{P}$ and each μ_i are continuous. Thus $\mathcal{X}_\alpha$ is closed, and hence compact since it is contained in a compact set. Finally, since f is continuous, we have $\mathcal{W}_\alpha = f(\mathcal{X}_\alpha)$ by Proposition 1.3, so $\mathcal{W}_\alpha$ is the image of a compact set under a continuous map and so is compact itself. ■

A special class of combination functions are what we will call the *idempotent mixed* combination functions.

3.3 Definition: A combination function $\mathcal{P}$ is said to be an *idempotent mixed combination function* if

1. $\mathcal{P}(\mu_1, \ldots, \mu_j, \ldots, \mu_n) \leq \mathcal{P}(\mu_1, \ldots, \mu_j', \ldots \mu_n) \quad \text{iff} \quad \mu_j < \mu_j'$ (monotonicity)
2. $\mathcal{P}(\mu, \ldots, \mu) = \mu \quad \text{for all} \quad \mu \in \mathrm{I}$ (idempotency)

The set of all such combination functions will be denoted $IMC(\mathrm{I}^n)$.

Examples of idempotent mixed combination functions are *min* itself, and $\mu_1^{\omega_1} \cdots \mu_n^{\omega_n}$ where $\omega_1 + \cdots + \omega_n = 1$. The following proposition gives an important property of idempotent mixed combination functions.

3.4 Proposition: *Let $\mathcal{P} \in IMC(\mathrm{I}^n)$. Then $\mathcal{P}(\mu_1, \ldots, \mu_n) \geq \min(\mu_1, \ldots, \mu_n)$.*

Proof: Define $\mu_* = \min(\mu_1, \ldots, \mu_n)$. Since $\mathcal{P}$ is monotonic, $\mathcal{P}(\mu_1, \ldots, \mu_n) \geq \mathcal{P}(\mu_*, \ldots, \mu_*)$. Since $\mathcal{P}$ is idempotent, $\mathcal{P}(\mu_*, \ldots \mu_*) = \mu_*$. Thus $\mathcal{P}(\mu_1, \ldots, \mu_n) \geq \mu_* = \min(\mu_1, \ldots, \mu_n)$. ■

Now the geometric hypotheses on the level sets of the function f and the combination function $\mathcal{P}$ will be presented. Recall from Section 2 the definitions of $\mathcal{Y}^{\alpha}_{min}$ and $\mathcal{Y}^{\alpha}_{max}$.

3.5 Definition: Let $\mathcal{P}$ be a combination function with corresponding membership function μ. Let $\mathcal{X}_{\alpha}$ denote the α-cut of μ. Let $f : \mathrm{R}^n \to \mathrm{R}$ be any function. We say that the pair $(\mathcal{P}, f)$ is *strongly inclusive* if:

$$\textbf{H1:} \quad \mathcal{X}_{\beta} \cap f^{-1}(\mathcal{Y}^{\alpha}_{min}) = \emptyset \quad \text{for all} \quad \beta \geq \alpha$$

and

$$\textbf{H2:} \quad \mathcal{X}_{\beta} \cap f^{-1}(\mathcal{Y}^{\alpha}_{max}) = \emptyset \quad \text{for all} \quad \beta \geq \alpha$$

and if f is continuous.

The name *strongly inclusive* is motivated by the following proposition:

3.6 Proposition: *Let $(\mathcal{P}, f)$ be strongly inclusive, and let μ be the membership corresponding to $\mathcal{P}$. The membership induced on* R *by f is denoted by μ_f. The α-cut for μ_f is denoted by $\mathcal{W}_{\alpha}$. Then $\mathcal{W}_{\alpha} \subseteq \mathcal{Y}_{\alpha}$.*

Proof: Let $y \in \mathcal{W}_{\alpha}$ so that $\mu_f(y) = \beta \geq \alpha$ where $\mu_f(y) = \sup\{\mu(x) \mid x \in \mathrm{R}^n, f(x) = y\}$.

We claim that there exists $x^* \in \mathcal{X}_{\beta}$ such that $f(x^*) = y$. Indeed, since f is continuous, $f^{-1}(y)$ is closed, so $f^{-1}(y) \cap \mathrm{supp}(\mu)$ is compact. Thus μ attains its supremum on $f^{-1}(y) \cap \mathrm{supp}(\mu)$ and this supremum must be β. Thus we can choose $x^* \in f^{-1}(y)$ to be a point where $\mu(x^*) = \beta$.

Now, since $\mu(x^*) = \beta \geq \alpha$, **H1** implies that $x^* \notin f^{-1}(\mathcal{Y}^{\alpha}_{min})$. Therefore $f(x^*) \geq y^{\alpha}_{min}$. Similarly, by **H2**, $x^* \notin f^{-1}(\mathcal{Y}^{\alpha}_{max})$ so $f(x^*) \leq y^{\alpha}_{max}$. So we have shown that $f(x^*) = y \in [y^{\alpha}_{min}, y^{\alpha}_{max}] = \mathcal{Y}_{\alpha}$. ∎

Therefore, under the hypotheses **H1** and **H2** and continuity of f, the ridge point induced α-cuts provide an upper bound for the actual α-cuts $\mathcal{W}_{\alpha}$. The name *strongly inclusive* comes from the fact that the opposite inclusion, $\mathcal{Y}_{\alpha} \subseteq \mathcal{W}_{\alpha}$ holds under much weaker hypotheses on $\mathcal{P}$ and f. This motivates the following definition.

3.7 Definition: Let $\mathcal{P}$ be a combination function, and let $f : \mathrm{R}^n \to \mathrm{R}$ be a function. We say that the pair $(\mathcal{P}, f)$ is *weakly inclusive* if $\mathcal{P} \in IMC(\mathrm{I}^n)$ and if f is continuous.

3.8 Proposition: *Let $(\mathcal{P}, f)$ be weakly inclusive, and let $\mathcal{W}_\alpha$ be the α-cut for the membership induced on* R *by f. Then $\mathcal{Y}_\alpha \subseteq \mathcal{W}_\alpha$.*

Proof: Let $y \in \mathcal{Y}_\alpha$.

We claim that there exists an $x \in \mathcal{X}_\alpha^1 \times \cdots \times \mathcal{X}_\alpha^n$ such that $f(x) = y$. Indeed, all the ridge points are contained in $\mathcal{X}_\alpha^1 \times \cdots \times \mathcal{X}_\alpha^n$ and therefore, in particular, x_{min}^α and x_{max}^α are so contained. Also note that $f(x_{min}^\alpha), f(x_{max}^\alpha) \in \mathcal{Y}_\alpha$. Since f is continuous and $\mathcal{X}_\alpha^1 \times \cdots \times \mathcal{X}_\alpha^n$ is connected, $f(\mathcal{X}_\alpha^1 \times \cdots \times \mathcal{X}_\alpha^n)$ is connected. Hence it must be either an open, closed, or half-open interval containing y_{min}^α and y_{max}^α. In any case we must have $\mathcal{Y}_\alpha \subseteq f(\mathcal{X}_\alpha^1 \times \cdots \times \mathcal{X}_\alpha^n)$ and so there is indeed an $x \in \mathcal{X}_\alpha^1 \times \cdots \times \mathcal{X}_\alpha^n$ such that $f(x) = y$.

If $\mathcal{X}_\alpha$ is the α-cut for the membership μ on R^n, then Proposition 3.4 implies that $\mathcal{X}_\alpha^1 \times \cdots \times \mathcal{X}_\alpha^n \subseteq \mathcal{X}_\alpha$. Therefore, the x found above must lie in $\mathcal{X}_\alpha$. Now, since $f(x) = y$ and $\mu(x) \geq \alpha$, $\mu_f(y) \geq \alpha$, hence $y \in \mathcal{W}_\alpha$. ■

Propositions 3.6 and 3.8 determine upper and lower bounds, respectively, for the α-cuts $\mathcal{W}_\alpha$ given various hypotheses on the combination function $\mathcal{P}$ and the function f. Now conditions will be stated which must be satisfied if the ridge point induced α-cuts provide an exact description of the actual α-cuts $\mathcal{W}_\alpha$. This is important for computational purposes since it immediately distinguishes cases when the vertex method *cannot* be used.

3.9 Proposition: *Let $\mathcal{P}$ be a combination function and let $f : \mathrm{R}^n \to \mathrm{R}$ be a function. If $\mathcal{W}_\alpha$ is the α-cut for the membership induced on* R *by f, and if $\mathcal{W}_\alpha = \mathcal{Y}_\alpha$ then the pair $(\mathcal{P}, f)$ satisfies* **H1** *and* **H2**.

Proof: Suppose that $\mathcal{W}_\alpha = \mathcal{Y}_\alpha$ and that **H1** does not hold. Then there exists $x \in \mathcal{X}_\beta \cap f^{-1}(\mathcal{Y}_{min}^\alpha)$ for some $\beta \geq \alpha$. Thus $\mu_f(f(x)) \geq \beta$ and $f(x) \in \mathcal{W}_\beta \subseteq \mathcal{W}_\alpha$. But, since $\mathcal{W}_\alpha = \mathcal{Y}_\alpha$, $f(x) \geq y_{min}^\alpha$. This contradicts $x \in f^{-1}(\mathcal{Y}_{min}^\alpha)$, therefore **H1** must hold.

Now suppose that $\mathcal{W}_\alpha = \mathcal{Y}_\alpha$ and that **H2** does not hold. Then there exists $x \in \mathcal{X}_\beta \cap f^{-1}(\mathcal{Y}^\alpha_{max})$ for some $\beta \geq \alpha$. Thus $\mu(x) \geq \beta$ so $\mu_f(f(x)) \geq \beta$, hence $f(x) \in \mathcal{Y}_\alpha$ since $\mathcal{Y}_\beta = \mathcal{W}_\beta \subseteq \mathcal{W}_\alpha = \mathcal{Y}_\alpha$. Therefore $f(x) \leq y^\alpha_{max}$, contradicting $x \in f^{-1}(\mathcal{Y}^\alpha_{max})$. Thus **H2** must hold. ■

The next proposition follows immediately from Propositions 3.6 and 3.8.

3.10 Proposition: *If the pair $(\mathcal{P}, f)$ is weakly inclusive, then $\mathcal{W}_\alpha = \mathcal{Y}_\alpha$ if and only if $(\mathcal{P}, f)$ is strongly inclusive.*

3.11 Remark: Since f is continuous, if the pair $(\mathcal{P}, f)$ is either strongly or weakly inclusive, is there an essential difference between the two notions with regard to the combination function $\mathcal{P}$? Indeed there is, as there are pairs $(\mathcal{P}, f)$ which are strongly inclusive but not weakly inclusive, and there are pairs $(\mathcal{P}, f)$ which are weakly inclusive but not strongly inclusive. Thus the relationship between the two notions is non-trivial. It would be interesting to explore a computational condition equivalent to that of being strongly inclusive, but such a condition appears to be somewhat complicated.

3.12 Remark: Observe that if two different pairs $(\mathcal{P}_1, f)$ and $(\mathcal{P}_2, f)$ are both strongly and weakly inclusive with the same f, then the resulting μ_f does not depend on the choice of $\mathcal{P}_1$ or $\mathcal{P}_2$. Thus identical induced membership functions can arise from different combination functions.

This is the case in the example presented in the introduction. In Figure 2, level sets of μ_f, μ'_f, and f are shown for $\alpha \in (0, 1)$. **H1** and **H2** are satisfied in this case, since level sets of f remain outside the β-cuts, for $\beta > \alpha$. Notice that if a different combination function were used, such as $\mu(x, y) = \mu(x)^{0.9999} * \mu(y)^{0.0001}$, then **H1** and **H2** would not be satisfied. Likewise, if f had different behavior near X_α, then again **H1** and **H2** would not be satisfied. Thus, it is pairs $(\mathcal{P}, f)$ which must be considered.

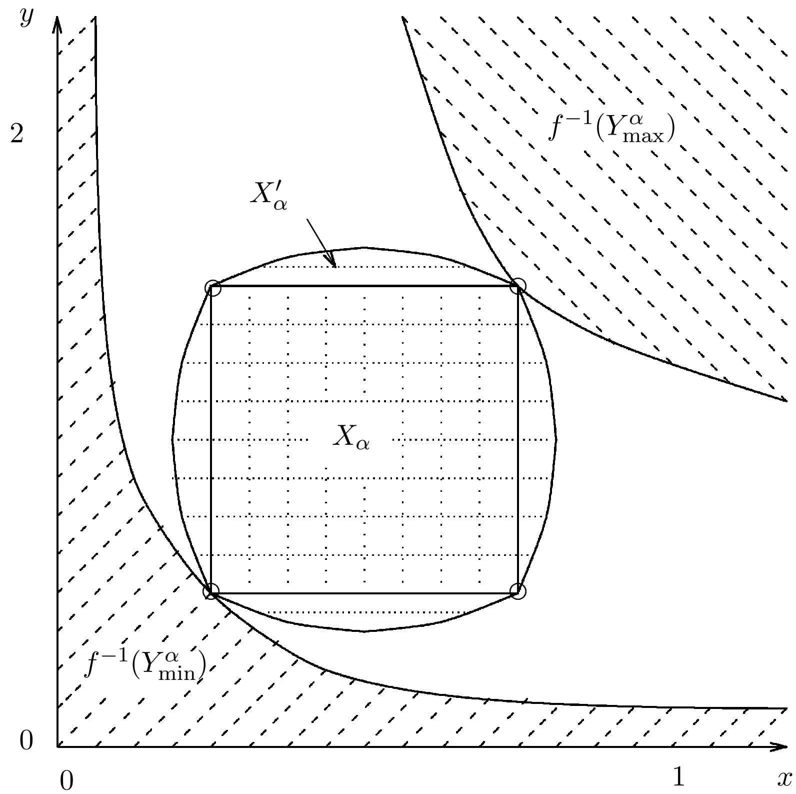

Figure 2: Levels sets of μ, μ', and f.

4 Conclusions

This paper presented geometric conditions which must be satisfied by a combination function $\mathcal{P}$ and a map f to enable the calculation of the induced membership by the endpoints of the level sets of each attribute membership. When the conditions are satisfied, the resulting induced membership is invariant between the $\mathcal{P}$ used and *min*.

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