

# Determining Optimal Points of Membership with Dependent Variables\*

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## Abstract

Consider sets  $\mathcal{X}$  and  $\mathcal{Y}$  each with associated membership functions. Further suppose that there is a map  $f : \mathcal{X} \rightarrow \mathcal{Y}$ . A membership may be induced on  $\mathcal{Y}$  from the membership of  $\mathcal{X}$  calculated using the extension principle, and the two memberships on  $\mathcal{Y}$  may be combined to give an overall membership. The point with the highest membership in  $\mathcal{X}$  can be determined directly or by back mapping the value with the highest membership in  $\mathcal{Y}$ . These two

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methods are equivalent under hypotheses on the membership functions and on  $f$ . An example calculation from engineering design is presented.

*Keywords:* Fuzzy numbers; algebra; engineering.

## Introduction

Consider a set  $\mathcal{X}$  equipped with a membership function: every point  $x \in \mathcal{X}$  has an associated value  $\mu(x) \in [0, 1] \subset \mathbb{R}$  representing the degree of membership for  $x$ . Suppose that another set  $\mathcal{Y}$  is mapped from  $\mathcal{X}$  by a function  $f$ , and that  $\mathcal{Y}$  itself has a membership function  $\nu(y) \in [0, 1] \subset \mathbb{R}$  independent of those on  $\mathcal{X}$ . Finally, consider combining all of the membership specifications  $\mu(x)$  and  $\nu \circ f(x)$  into an *overall membership*  $\mu_c(x)$  on  $\mathcal{X}$ . This paper demonstrates that points in  $\mathcal{X}$  which maximize overall membership can also be determined by another method. One can induce the membership specified for  $\mathcal{X}$  onto  $\mathcal{Y}$  by the extension principle. There one can find the  $y \in \mathcal{Y}$  which maximize the overall membership, and then back map to  $\mathcal{X}$  simply by looking up the values of  $x$  used in the original forward mapping to the optimal  $y$ . This is true even though the induced membership on  $\mathcal{Y}$  involves only the membership  $\mu$ , and does not consider any of the dependent set membership  $\nu$ . The results, however, are the points which maximize the overall membership  $\mu_c$ : the supremum of the combination of the membership specified both on  $\mathcal{X}$  and  $\mathcal{Y}$ .

As motivation for this work, consider engineering design problems. In a typical engineering problem solved using imprecise variables, the imprecision can arise from the inability to specify the final precise values of the design variables (to be used in the problem model). We use the term *design parameter* to denote these variables which describe the engineering model, and must have precise values determined for the model to be solved. The initial lack of precision can be represented formally using fuzzy sets, and hence we have an example of a set  $\mathcal{X}$  with a membership function.

Dependent variables are those which are functionally related to the design parameters, and give indications of performance of the model. We term these variables *performance parameters*. These form a dependent variable space  $\mathcal{Y}$ . There are also membership specifications, independent of those specified on  $\mathcal{X}$ , placed on  $\mathcal{Y}$ .

Typically, all of the imprecise quantities are combined to solve the complete model; that is, select precise values for the design parameters [5]. This is calculated by maximizing the overall membership over the set of design parameters  $\mathcal{X}$ . Examples are given in [5].

However, it is usually of considerable interest to observe the design parameter membership expressed on the performance parameter space. This can be calculated using the extension principle [8], as will be discussed subsequently. Doing so allows a comparison of the membership functions specified on the design parameters with the membership functions specified on the dependent performance parameters. Discussion and examples are given in [6, 7].

However, this calculation can be computationally difficult. This paper demonstrates that this difficulty can be reduced, since the extension principle calculation will allow one to determine the final design parameter values directly, rather than through another computationally difficult search. We will show that the result of optimizing overall membership over the design parameters is the same as optimizing overall membership over the dependent performance parameters, and then back mapping the result to the design parameters by the image of the extension principle, even though this original inducement of membership does not involve the membership specified on the dependent performance parameters.

This work is an extension of the idea, originally proposed independently by both Dubois [1] and Wood and Antonsson [6], of propagating memberships from independent variables to dependent variables, selecting optimal performance, and mapping back the membership level so as to determine the optimal parameters. We extend these ideas to general sets, rather than  $\mathcal{X} \simeq \mathbb{R}^N$  and  $\mathcal{Y} \simeq \mathbb{R}$ , and to more general connectives than *min*. We consider problems which have possibly more than one optimal solution. Further, we consider independent membership specifications  $\nu$  on  $\mathcal{Y}$ , not considered in either of these previous works.

## 1 Definitions and Notation

The results desired do not depend on any structure associated with the variables used; as such, in the following sections, we will be working with general sets and membership functions. Thus

**1.1 Definition:** A *membership function* on a set  $\mathcal{X}$  is a function  $\mu : \mathcal{X} \rightarrow \mathbb{I} = [0, 1] \subset \mathbb{R}$ .

We are also interested in membership induced on a set from the membership on another set. This is typically done via a map between the sets as in Zadeh [8], see [2, 3] for a review.

**1.2 Definition:** Let  $\mathcal{X}$  and  $\mathcal{Y}$  be sets and let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a function. The *induced membership* on  $\mathcal{Y}$  is

$$\nu_f(y) = \sup\{\mu(x) \mid x \in \mathcal{X}, f(x) = y\}$$

We adopt the convention that  $\nu_f(y) = 0$  if  $f^{-1}(y) = \emptyset$ . Note that the induced membership function need not be continuous, even if  $\mathcal{X}$  and  $\mathcal{Y}$  are topological spaces with continuous membership functions and  $f$  is a continuous map.

Now we wish to combine the membership functions on  $\mathcal{X}$  and  $\mathcal{Y}$  in the following manner. Let  $\mathcal{P} : \mathbb{I}^2 \rightarrow \mathbb{I}$  be monotonic in its first argument. Thus, if  $0 \leq a \leq b \leq 1$ , then  $\mathcal{P}(a, t) \leq \mathcal{P}(b, t)$  for all  $t \in \mathbb{I}$ . In this manner we can define

$$\mu_c(x) = \mathcal{P}(\mu(x), \nu \circ f(x)) \quad (1.1)$$

and

$$\nu_c(y) = \mathcal{P}(\nu_f(y), \nu(y)) \quad (1.2)$$

as membership functions on  $\mathcal{X}$  and  $\mathcal{Y}$ , respectively. Thus  $\mu$  represents membership on the design parameters, and  $\nu$  represents membership on the performance parameters. The objective is to determine the *most preferred point* in  $\mathcal{X}$ ; that is, the point in  $\mathcal{X}$  which maximizes the overall membership  $\mu_c$ .

Since  $\mu_c$  and  $\nu_c$  are membership functions, and hence bounded, they have finite supremum over  $\mathcal{X}$  and  $\mathcal{Y}$ , respectively. So we can define the following numbers in  $\mathbb{I}$ .

$$\|\mu_c\|_\infty = \sup\{\mu_c(x) \mid x \in \mathcal{X}\} \quad (1.3)$$

and

$$\|\nu_c\|_\infty = \sup\{\nu_c(y) \mid y \in \mathcal{Y}\} \quad (1.4)$$

We will further assume that the sets

$$\mathcal{X}^* = \{x \in \mathcal{X} \mid \mu_c(x) = \|\mu_c\|_\infty\} \quad (1.5)$$

and

$$\mathcal{Y}^* = \{y \in \mathcal{Y} \mid \nu_c(y) = \|\nu_c\|_\infty\} \quad (1.6)$$

are not empty. This is true, for example, when  $\mathcal{X}$  and  $\mathcal{Y}$  are topological spaces, and  $\mu_c$  and  $\nu_c$  are continuous with compact support. However, it may be true in other cases, and we do not wish to exclude these from consideration.

## 2 Results

The first result we will state will form the foundation of all our results to follow.

**2.1 Proposition:**  $\nu_c(y) = \sup\{\mu_c(x) \mid x \in \mathcal{X}, f(x) = y\}$

*Proof:* We will first prove a technical lemma.

**2.2 Lemma:** *Let  $\mathcal{Q} : \mathbb{R} \rightarrow \mathbb{R}$  be monotonic, and let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be any function.*

*Then*

$$\mathcal{Q}(\sup\{g(x) \mid x \in \mathbb{R}\}) = \sup\{\mathcal{Q}(g(x)) \mid x \in \mathbb{R}\}$$

*Proof:* Since  $Q$  is monotonic, the expression on the right-hand side of the equality will be evaluated at the value of  $x$ , possibly  $\pm\infty$ , where  $g$  attains its supremum. But this is precisely what the left-hand side of the equation returns.

Now we proceed with the proof of the proposition. Using 2.2 we have

$$\begin{aligned}
\nu_c(y) &= \mathcal{P}(\nu_f(y), \nu(y)) \\
&= \mathcal{P}(\sup\{\mu(x) \mid x \in \mathcal{X}, f(x) = y\}, \nu(y)) \\
&= \sup\{\mathcal{P}(\mu(x), \nu(y)) \mid x \in \mathcal{X}, f(x) = y\} \\
&= \sup\{\mathcal{P}(\mu(x), \nu \circ f(x)) \mid x \in \mathcal{X}, f(x) = y\} \\
&= \sup\{\mu_c(x) \mid x \in \mathcal{X}, f(x) = y\}
\end{aligned}$$

The following lemma will be useful in the sequel.

**2.3 Lemma:**  $\|\mu_c\|_\infty = \|\nu_c\|_\infty$

*Proof:* Choose  $y \in \mathcal{Y}^*$  so that  $\nu_c(y) = \|\nu_c\|_\infty$ . But  $\nu_c(y) = \sup\{\mu_c(x) \mid x \in \mathcal{X}, f(x) = y\} \leq \|\mu_c\|_\infty$ . Thus  $\|\nu_c\|_\infty \leq \|\mu_c\|_\infty$ . Now let  $x \in \mathcal{X}^*$  so that  $\mu_c(x) = \|\mu_c\|_\infty$ . Then  $\nu_c(f(x)) = \sup\{\mu_c(x') \mid f(x') = f(x)\} \geq \|\mu_c\|_\infty$ . Thus  $\|\nu_c\|_\infty \geq \|\mu_c\|_\infty$  so  $\|\nu_c\|_\infty$  must equal  $\|\mu_c\|_\infty$ .

We are interested in ascertaining the peak membership values in  $\mathcal{X}$  given those in  $\mathcal{Y}$ .

**2.4 Theorem:**  $f(\mathcal{X}^*) \subset \mathcal{Y}^*$ . Conversely, if  $\mathcal{X}$  and  $\mathcal{Y}$  are topological spaces, and if

- i)  $f$  is continuous, and has compact support, and
  - ii) if  $\mu_c$  is continuous when restricted to  $f^{-1}(y)$  for each  $y \in \mathcal{Y}^*$
- then  $\mathcal{Y}^* \subset f(\mathcal{X}^*)$ .

*Proof:* Let  $x \in \mathcal{X}^*$ . Then  $\mu_c(x) = \|\mu_c\|_\infty$ . Then  $\nu_c(f(x)) = \sup\{\mu_c(x') \mid f(x') = f(x)\} \geq \|\mu_c\|_\infty$ . By 2.3  $\nu_c(f(x)) = \|\nu_c\|_\infty$  and so  $f(x) \in \mathcal{Y}^*$ .

Now suppose i) and ii) hold, and let  $y \in \mathcal{Y}^*$  so that  $\nu_c(y) = \|\nu_c\|_\infty$ . Since  $f$  is continuous,  $f^{-1}(y)$  is closed for each  $y \in \mathcal{Y}^*$  (since such a  $y$  is itself closed). Thus, since  $f$  has compact support,  $f^{-1}(y)$  is also compact (a closed subset of a compact set is also compact). Now  $\mu_c$  is continuous on the compact set  $f^{-1}(y)$  and so assumes its maximum value in the set. But, by definition,  $\sup\{\mu_c(x) \mid x \in f^{-1}(y)\} = \|\nu_c\|_\infty$ , so there is some  $x \in f^{-1}(y)$  such that  $\mu_c(x) = \|\nu_c\|_\infty = \|\mu_c\|_\infty$ , so  $x \in \mathcal{X}^*$ . Thus  $\mathcal{Y}^* \subset f(\mathcal{X}^*)$ .

In practice, if  $f$  is continuous, it may be possible to satisfy the hypothesis of compact support simply by restriction to the domain of interest.

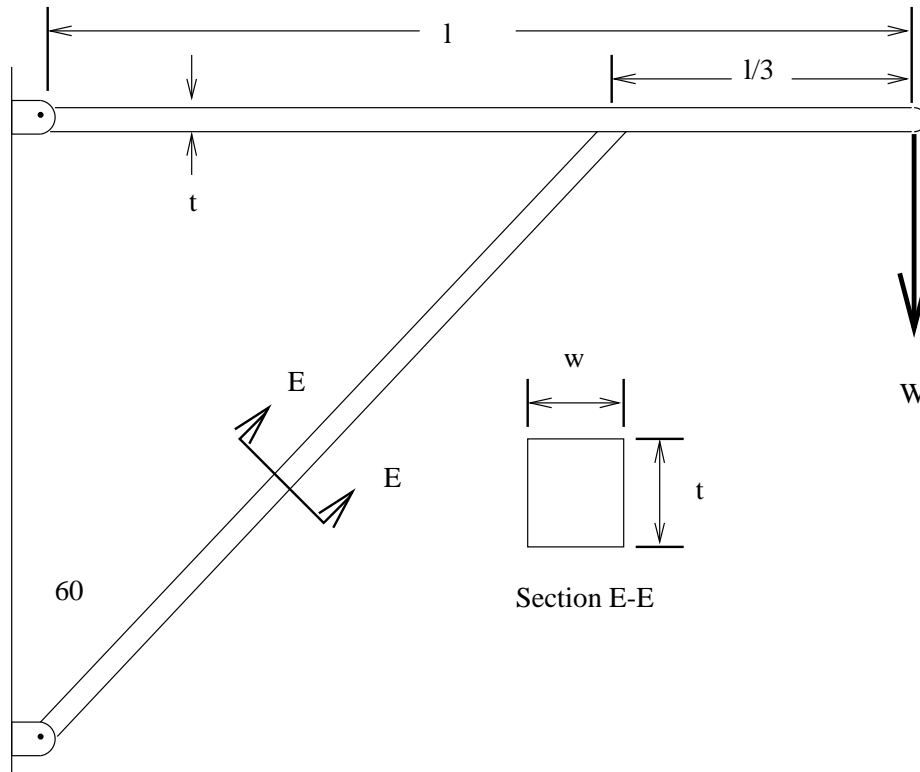


Figure 1: Design Example: Structural Frame.

### 3 Example: Imprecise Engineering Calculations

Consider the design of a structural frame intended to support a weight  $W$  a distance  $l$  from a wall, as shown in Figure 1. The width and thickness of the beam is  $w$  and  $t$  respectively. Given this configuration, a designer may have preferences for the various parameters based on geometric constraints, customer requirements, etc. For the purposes of this illustration, we shall assume that all of these *design parameters* are crisp, except for  $w$  and  $t$ , which the designer has given imprecise specifications for, as shown in Figure 2.

Given a configuration (as shown in Figure 1), typically a designer performs calculations to rate different values of the design parameters. In this example, a typical *performance parameter* might be maximum bending stress in the horizontal

bar. This is given by

$$\begin{aligned} \sigma &: \mathbb{R}^2 \rightarrow \mathbb{R} \\ (w, t) &\mapsto \frac{2l(W + \frac{\rho q w t l}{6})}{w t^2} \end{aligned}$$

The designer must ensure that the bending stress is not excessive; as such, there is an imprecise specification on the dependent performance parameter space  $\mathbb{R}$ , as shown in Figure 2. The remaining parameter values are shown in Table 1.

Now given these specifications, the designer can perform calculations, such as observing how the membership specifications made on  $w$  and  $t$  induce membership on the performance parameter space. This will be calculated using the extension principle, as reflected by Definition 1.2 (with  $\mathcal{X} = \mathbb{R}^2$ , and  $\mathcal{Y} = \mathbb{R}$ ).

The results of Section 2 apply to general combination functions. We will consider two different cases. The first is the traditional fuzzy set combination of  $\mathcal{P}(a, b) = \min\{a, b\}$ . Also, we consider  $\mathcal{P}(a, b) = a^{\frac{2}{3}} \cdot b^{\frac{1}{3}}$ .

First, the membership  $\mu$  on  $\mathcal{X}$  must be defined. For the two cases considered:  $\mu_1(w, t) = \min\{\mu_w(w), \mu_t(t)\}$ , and  $\mu_2(w, t) = (\mu_w(w) \cdot \mu_t(t))^{\frac{1}{2}}$ . Given these definitions, the induced membership  $\nu_f$  is defined for both combinations and is shown in Figure 3. An interesting feature of this problem is that both combination functions  $\mu_1$  and  $\mu_2$  result in the exact same membership  $\nu_f$  on  $\sigma$ , as calculated using the extension principle, Definition 1.2. This invariance is a result of particular geometric conditions which the two combination functions and the function  $\sigma$  satisfy, as discussed in [4].

Having performed this calculation, the designer can then compare this induced membership with the requirement membership  $\nu$  as shown in Figure 2. For example, the designer may select the optimal value of  $\sigma$  by combining the two membership functions with  $\nu_{1,c}(\sigma) = \min\{\nu_f(\sigma), \nu(\sigma)\}$ , or, perhaps with a combination  $\nu_{2,c}(\sigma) = \nu_f(\sigma)^{\frac{2}{3}} \cdot \nu(\sigma)^{\frac{1}{3}}$ . These reflect two different possible combination functions, but the results of Section 2 apply to both. The results of applying the two methods are shown in Figure 4.

Consider now the optimal values of  $\sigma$  as defined by the maximum of values of  $\nu_{1,c}$  and  $\nu_{2,c}$ . For  $\nu_{1,c}$ , the optimal value is  $\sigma_1^* = 0.298$  (GPa). For  $\nu_{2,c}$ , the optimal value is  $\sigma_2^* = 0.295$  (GPa). These define the most preferred values of performance. The question is what are the optimal points  $w_i^*$  and  $t_i^*$  which produce these performances  $\sigma_i^*$ , for  $i = 1, 2$ .

It can be shown that the hypotheses *i*) and *ii*) of 2.4 hold. Therefore the points used in the extension principle calculation of  $\nu_f(\sigma^*)$  are in fact the points used to

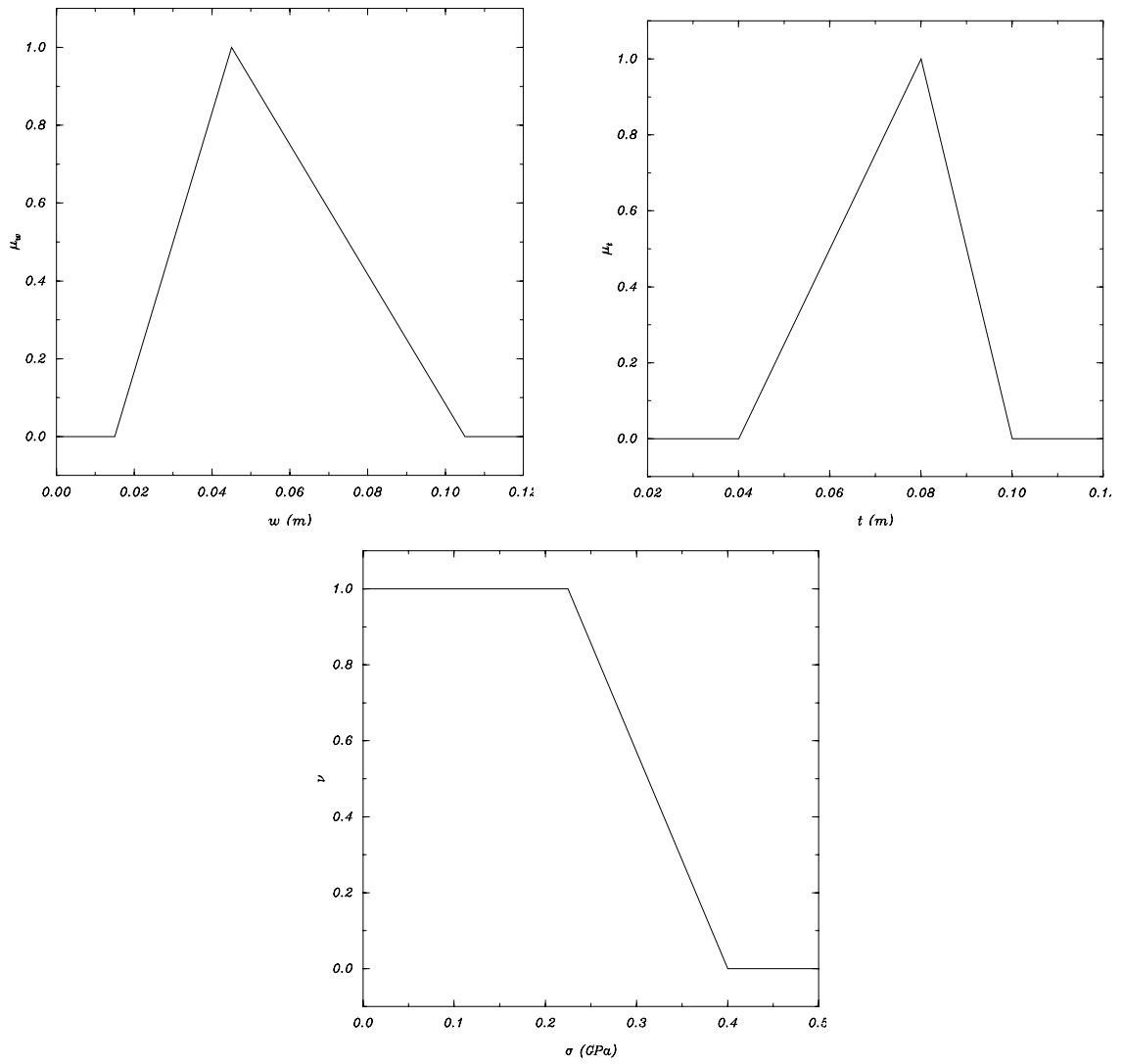


Figure 2: Specified membership functions.

Table 1: Constant Values

parameter	Value
W	20 <i>kN</i>
l	4 <i>m</i>
g	9.8 $\frac{m}{s^2}$
$\rho$	7830 $\frac{kg}{m^3}$

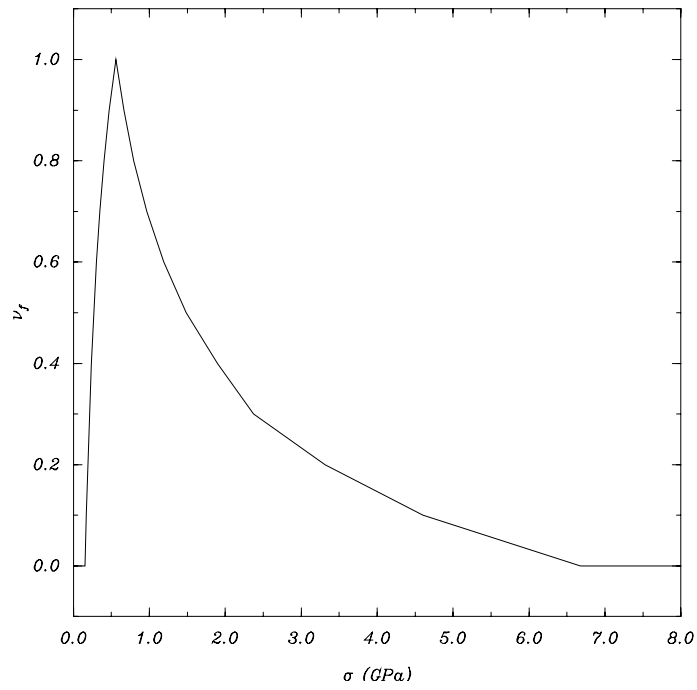


Figure 3: Induced membership.

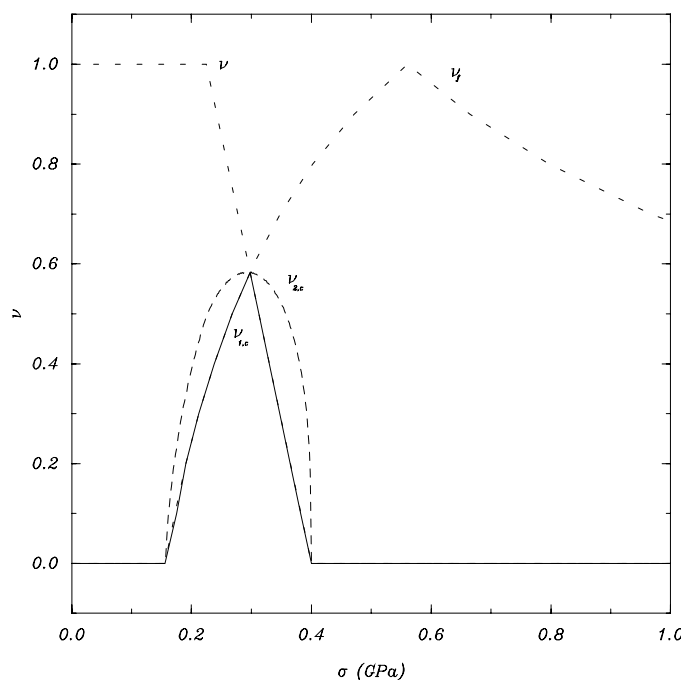


Figure 4: Overall membership on  $\sigma$ .

compute  $\nu_c(\sigma^*)$ . That is, when using *min* as a combination, at the optimal value of performance  $\sigma_1^* = 0.298$  (*GPa*), the points  $w_1 = 0.07003$  (*m*) and  $t_1 = 0.08834$  (*m*) were used to calculate  $\nu_f(\sigma_1^*)$ . As well, when using the product as a combination, at the optimal value  $\sigma_2^* = 0.295$  (*GPa*), the points  $w_2 = 0.07697$  (*m*) and  $t_2 = 0.08471$  (*m*) were used to calculate  $\nu_f(\sigma_2^*)$ . Theorem 2.4 demonstrates that these values of  $w_i$  and  $t_i$  are the optimal points of overall membership  $\mu_{i,c}$ , for  $i = 1, 2$ . That is, if the designer had as well calculated  $w_i^*$  and  $t_i^*$  defined by

$$\mu_{1,c}(w_1^*, t_1^*) = \sup\{\min\{\mu_w(w), \mu_t(t), \nu \circ \sigma(w, t)\} \mid (w, t) \in \mathbb{R}^2\}$$

or

$$\mu_{2,c}(w_2^*, t_2^*) = \sup\{(\mu_w(w) \cdot \mu_t(t) \cdot \nu \circ \sigma(w, t))^{\frac{1}{3}} \mid (w, t) \in \mathbb{R}^2\}$$

respectively (reflecting the two different combination functions  $\nu_c$  on  $\sigma$ ), then the resulting  $w_i^* = w_i$  and  $t_i^* = t_i$ . Notice the key aspect of this:  $\nu_f$  is used to back-map the optimal points in  $\mathcal{X}$  of  $\nu_c$ . Thus one can use the pre-image of the design parameter induced membership  $\nu_f$  to find the optimal point of overall membership  $\mu_c$ , even though  $\nu_f$  does not involve the performance parameter membership specification  $\nu$ .

## Conclusion

This paper presents an observation about membership optimization problems with dependent parameters: one can maximize membership over either the set  $\mathcal{X}$ , or over the dependent set  $\mathcal{Y}$ . One could compose the dependent set membership onto  $\mathcal{X}$  and maximize overall membership over  $\mathcal{X}$  to find the optimal point in  $\mathcal{X}$ . Alternatively, one could induce the original set membership onto  $\mathcal{Y}$  and maximize overall membership over  $\mathcal{Y}$ , and then use the pre-image of the induced membership to determine the optimal point in  $\mathcal{X}$ . This is true despite the fact that the pre-image is of the induced membership only, not the pre-image of the overall membership.

This fact is useful, since in imprecise engineering design, it is very beneficial to observe the induced membership of the design parameters on the performance parameter space, to observe the performance achievable in the model. This technique is superior to others, since it presents visual information to the designer about the model. An optimization routine will produce a (hopefully globally optimal) solution, but it presents no real information about critical aspects in the problem. If this first design process is adopted, it is useful to know the induced membership pre-image of the optimal performance value is the optimal membership point, since this eliminates the need for a subsequent search.

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